Physics 200-05
Assignment 7

1. Show explicitly that the eigenvalues for the matrix $A = a_0 I + \vec{a}\cdot\vec{\sigma}$ are $a_0 \pm \sqrt{\vec{a}\cdot\vec{a}}$.

If

$$a_1 = a\sin(\theta)\cos(\phi) \tag{1}$$
$$a_2 = a\sin(\theta)\sin(\phi) \tag{2}$$
$$a_3 = a\cos(\theta) \tag{3}$$

then show that the eigenvector for the $a_0 + a$ eigenvalue is

$$|a_0 + a\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \tag{4}$$

and for the other eigenvalue the eigenvector is

$$|a_0 - a\rangle = \begin{pmatrix} -e^{-i\phi}\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix} \tag{5}$$

2. Consider the state vector

$$|\psi\rangle = \frac{1}{\sqrt{(2)}}\begin{pmatrix} 1 \\ \frac{1+i}{\sqrt{2}} \end{pmatrix} \tag{6}$$

a)What is the unit vector $|\phi\rangle$ orthogonal to this vector? Ie, $\langle\phi||\psi\rangle = 0$?

b) Show that the matrix $A = |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$ has eigenvalues ± 1 and eigenvectors $|\psi\rangle$ and $|\phi\rangle$. (Remember that $|\mu\rangle\langle\nu|$ is the product of a column vector times a row matrix, which is a 2x2 matrix if the $|\mu\rangle$ and $|\nu\rangle$ are 1x2 vectors.)

$$\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} \tag{7}$$

Finally show that $|\psi\rangle\langle\psi|$ is a projection operator (has a single eigenvalue of value 1 and the other eigenvalue has value 0) with $|\psi\rangle$ as the eigenvector with 1 as the eigenvalue.

3. Given the matrix

$$A = \begin{pmatrix} 3 & 2+2i \\ 2-2i & -1 \end{pmatrix} \tag{8}$$

what are the values of $a_0,\ a_1,\ a_2,\ a_3$ and what are the eigenvalues of this matrix?

What is the projection matrix onto the larger eigenvalue? If the state $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ what is the probability that the largest eigenvalue of A is obtained in a measurement.

4) Show that

$$[A, BC] = [A, B]C + B[A, C] \tag{9}$$

where $A,\ B,\ C$ are matrices and $[A, B] = AB - BA$ is the commutator.

Show that if X and P obey

$$[X, P] = i\hbar I \tag{10}$$

and if we define the Energy as

$$H = \frac{1}{2m}P^2 + \frac{k}{2}X^2 \tag{11}$$

where m and k are real numbers. Then

$$[X, H] = i\hbar\frac{1}{m}P \tag{12}$$

and

$$[P, H] = -i\hbar kX \tag{13}$$

Show that if we define the non-Hermitean matrix

$$A = (km)^{\frac{1}{4}}X + i\frac{1}{(km)^{\frac{1}{4}}}P$$

, then

$$[A, H] = \hbar\sqrt{\frac{k}{m}}A \tag{14}$$

Finally, show that if $|E\rangle$ is an eigenvector if H with eigenvalue E, then $A|E\rangle$ is an eigenvector of H with eigenvalue $E - \hbar\sqrt{\frac{k}{m}}$. A is called the annihilation operator for the simple harmonic oscillator because it annihilates one unit of energy. (Ie, if the state has energgy E, the new state after operating on it by A has one unit less energy)

This can be used to show that the eigenvalues for the Harmonic oscillator must have values $(n + \frac{1}{2})\hbar\sqrt{\frac{k}{m}}$ where n is a positive integer. (You do not have to show this, but if you want to do it for yourself, The key is that there must be a minimum eigenvalue since $\langle\psi|H|\psi\rangle$ is greater than 0 and thus the A cannot step the eigenvalues to less than 0.)

(See the text book for further explication.)