Review

I. FIELDS

A field is a function space and time. Ie, for every point in space and in time, a field has a value, a value which is usually different for each point in space and time. In classical particle mechanics, a particle has a position, a point in space and which point in space it is at is a function of time. All quantities like $\vec{E}, \vec{B}, \Phi, \vec{A}, \vec{J}, \rho, \Lambda$ are fields. They have a value at each point in space and time. Those values can be just single numbers (like ρ or tangent vectors in space (little arrows pointing in some direction in space) or cotangent vectors (exemplified by the slope of functions at that point). In the following, the vectors (like $\vec{A}$ or $\vec{B}$, or A_i of B^i may be written without the $\vec{}$ or the super or subscript allowing just the symbol to carry the meaning. And it may have no functional form ($\vec{E}(t, x, y, z)$ could d just be written as E with the rest understood.

II. VECTORS, TENSORS AND COORDINATES:

Two types of vectors: tangent vectors to curve, and gradient vectors of functions. They differ by how they transform under a change of coordinates

$$\frac{d\tilde{x}^i}{d\lambda} = \frac{\partial \tilde{x}^i}{x^j}\frac{dx^j}{d\lambda} \tag{1}$$

$$\frac{\partial f(x(\tilde{x}))}{\partial \tilde{x}^i} \equiv \partial_{\tilde{j}} f = \partial_{\tilde{i}} x^j \partial_j f \tag{2}$$

Anything that transforms like a tangent vector is called a tangent type object, while something that transforms like a gradient is a cotangent type object.

Summation convention. If one has two indices, one up (tangent type) and one down (cotangent or gradient type) with the same "name" (ie index label) , then one sums over that index

$$V^i W_i \equiv \sum_{i=1}^{3} V^i W_i \tag{3}$$

Since under change of coordinates,

$$\tilde{V}^j \tilde{W}_j = (V^i \partial_i \tilde{x}^j)(\partial_{\tilde{j}} x^k W_k) \tag{4}$$

$$= V^i (\partial_i \tilde{x}^j \partial_{\tilde{j}} x^k) W_k = V^i \delta_i^k W_k = V^i W_i \tag{5}$$

Tensor

An object with indices each of whose up indices transform like a tangent vector, and lower like a cotangent vector

$$T^{ij}{}_k{}^l = \tilde{T}^{stv}_u (\partial_{\tilde{s}} x^i)(\partial_{\tilde{t}} x^j)(\partial_k \tilde{x}^u)(\partial_{\tilde{v}} x^l) \tag{6}$$

Beware

$$\partial_i V^j = \partial_i \tilde{x}^k \partial_{\tilde{k}} \left(\partial_{\tilde{l}} x^j \tilde{V}^l \right) + \partial_i \tilde{x}^k \partial_{\tilde{k}} \partial_{\tilde{l}} x^j \partial_{\tilde{l}} x^j V^j \tag{7}$$

The first term is the transformation one would expect of a tensor, while the second term contains a derivative of the Jacobean, and this is NOT a tensor transformation. Derivatives need to be redefined to become tensor derivatives, in such a way as to get rid of that second term.These are covariant derivatives which are not covered in this course, except in the case of the divergence.

Metric g_{ij}. $g_{ij} V^i V^j$ is the length of the vector V^i. Symmetric $g_{ij} = g_{ji}$

Inverse metric g^{ij} is the inverse of metric $g_{ij} g^{jk} = \delta_i^k$

Raising and lowering indices $W^i = g^{ij} W_j$, $V_i = g_{ij} V^j$

Divergence

$$\text{Divergence}V^i = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}V^i \tag{8}$$

where g is the determinant of the matrix g_{ij}. The $\sqrt{g}$ are there to compensate for the coordinate transformation of the tangent vector under a coordinate transformation.

Anti-symmetric tensor ϵ_{ijk} becomes negative under any swapping of two indices. Eg,

$$\epsilon_{kji} = -\epsilon_{ijk} \tag{9}$$
$$\epsilon_{123} = \sqrt{g} \tag{10}$$

Cross product

$$A \times B \to \epsilon_{ijk}A^jB^k \tag{11}$$
$$\text{or} \tag{12}$$
$$U \times V = \epsilon^{ijk}U_jV_k \tag{13}$$

Curl

$$\nabla \times A \to \epsilon^{ijk}\partial_j A_k \tag{14}$$

Note that the derivative does not hit the $1/\sqrt{g}$ in ϵ^{ijk}.

Poisson's equation

$$\nabla^2\Phi \to \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ik}\partial_k\Phi \tag{15}$$

It is the divergence of the index-raised gradient of Φ.

Cartesian coordinates: Coordinates such that

$$g_{ij} = \delta_{ij} \tag{16}$$

III. SOLUTIONS

Gauss's Thm:

Volume element:

$$d^3\mathfrak{V} = \sqrt{g}(\mathfrak{X})d^3x \tag{17}$$

Surface element:

If coordinates are chosen so that x^1 is perpendicular to the surface just at each point in the surface, and the surface is defined by x^1 =const, and x^1 increases as you go through the surface, then

$$d^2\mathfrak{S} = \sqrt{g_{11}}\sqrt{^2g}dx^2dx^3 \tag{18}$$

where ^2g is the two-dimensional metric inside the metric. x^1, x^2, x^3 must be a right handed coordinate system, and (thumb pointing along increasing x^1, first finger along increasing x^2, second finger along increasing x^2 all of the right hand, and all angles between increasing coordinates less then 180 degrees.

Gauss's Thm:

$$\int_{\mathfrak{V}} \frac{1}{\sqrt{g}}\partial_i\sqrt{g}E^i d^3\mathfrak{V} = \int_{\mathfrak{S}} E^1 d^2\mathfrak{S}. \tag{19}$$

Or

$$\int \nabla \tag{20}$$
$$cdot\vec{E}d^3\mathfrak{V} = \int \vec{n}\cdot E\sqrt{^2g}dx^2dx^3 \tag{21}$$

where $\vec{n}$ is a unit cotangent vector perpendicular to the surface.

Stokes's Thm

The integral of the curl of a vector over a surface equals the integral of the vector over the line defining the boundary to the surface. Let the surface be defined by x^1 = const, and the coordinate chosen so that the lines of constant x^2, x^3 hit the surface perpendicular to the surface, and the line defined by x^2 constant, and x^2 axis perpendicular to the line where it meets the line, then

$$\int \epsilon^{1ij}\partial_i W_j \sqrt{g_{11}}\sqrt{^2g}dx^2dx^3 = \int W_3 d(x^3) \tag{22}$$

or

$$\int\int \nabla \times \vec{B} \cdot nd^2x = \int \vec{B} \cdot d\vec{l} \tag{23}$$

Maxwell's equations

$$\begin{aligned}
\vec{E} = -\nabla\Phi - \partial_t\vec{A}; &\quad \vec{B} = \nabla \times \vec{A} \\
\nabla \times \vec{E} = \partial_t\vec{B}; &\quad \nabla \cdot \vec{B} = 0 \\
\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho; &\quad \nabla \times \vec{B} = \frac{1}{c^2}\partial_t\vec{E}^2 + \mu^0\vec{J} \\
\partial_t\rho + \nabla \cdot \vec{J} = 0 & \\
\epsilon_0\mu_0 = \frac{1}{c^2}; &\quad \mu_o = 4\pi 10^{-7}; \quad c = 299,792,458 m/s(defined).
\end{aligned} \tag{24}$$

Gauge invariance

$$\vec{A} \to\to \vec{A} + \nabla\Lambda; \quad \Phi \to \Phi - \partial_t\Lambda \tag{25}$$

leaves $\vec{B}, \vec{E}, \rho, \vec{J}$ the same functions (they are gauge invariant).

Statics:

If the inclusion of time derivatives makes negligible difference to the solution of problem, then you are dealing with Statics. Usually means that one can set all time derivatives to zero, but sometimes "secular effects" (integral over time) can be non-negligible.

Energy:

$$\mathcal{E} = \frac{1}{2}\int \epsilon_0\vec{E}\vec{E} + \frac{1}{\mu^0}\vec{B} \cdot \vec{B})d^3\mathfrak{V} \tag{26}$$

$$\tag{27}$$

Interaction energy:

$$\int \mathcal{E}_{\mathcal{I}} d^2\mathfrak{V} = \int \epsilon_0\vec{E}_a\vec{E}_b + \frac{1}{\mu^0}\vec{B}_a \cdot \vec{B} * b)d^3\mathfrak{V} \tag{28}$$

$$= \int \rho_a\vec{\Phi}_b + \vec{J}_a \cdot \vec{A}_b d^2\mathfrak{V} \tag{29}$$

Energy and Momentum Conservation

Energy:

$$\partial_t\frac{1}{2}(\epsilon_0\vec{E} \cdot \vec{E} + \frac{1}{\mu_0}\vec{B} \cdot \vec{B}) + \vec{J} \cdot \vec{E} + \nabla \cdot (\epsilon_0\vec{E} \times \vec{B}) = 0 \tag{30}$$

Momentum

$$\partial_t(\frac{1}{\mu_0}\vec{E} \times \vec{B}) + (\rho\vec{E} + \vec{J} \times \vec{B}) + \nabla \cdot \overset{\leftrightarrow}{\Theta} = 0 \tag{31}$$

where $\vec{J}\cdot\vec{E}$ is the energy fed by the electromagnetic field into the matter carrying the current, $(\rho\vec{E}+\vec{J}\times\vec{B})$ is the force on the matter field, and the tensor $\overset{\leftrightarrow}{\Theta}$ is given by

$$\Theta^{ij}=\epsilon_0(E^iE^j-\frac{1}{2}E^kE_k\delta^{ij})+\frac{1}{\mu^0}(B^iB^j-\frac{1}{2}B^kB_k) \tag{32}$$

and is the stress tensor of the electromagnetic field.

Note that from the energy equation, only the E field can feed energy from the electromagnetic field into matter, or extract energy from the matter into the electromagnetic field, but both the magnetic and the electric fields can exert forces (feed momentum changes) on the matter.

IV. SOLUTIONS

$$\nabla^2\Phi=-\frac{1}{\epsilon_0}\rho \tag{33}$$

$$\Phi=-\frac{1}{\epsilon_0}\int G(\vec{x},\vec{x}')\rho(x')d^3x' \tag{34}$$

.For free space

$$G(\vec{x},\vec{x}')=\frac{1}{4\pi\sqrt{(x^1-x'^1)^2-(x^2-x'^2)^2-(x^3-x'^3)^2}} \tag{35}$$

$$\nabla^2G(x,x')=\delta(x^1-x'^1)^2-\delta(x^2-x'^2)\delta(x^3-x'^3) \tag{36}$$

For a space with boundaries,we have

$$G_B(\vec{x},\vec{x}')=G(\vec{x},\vec{x}')+G_F(\vec{x},\vec{x}') \tag{37}$$

where G_F solves the free equation $\nabla^2G_F=0$ so that the appropriate boundary condition is set.

Spherical coords:

$$\frac{1}{r^2}\partial_r(r^2\partial_r\Phi(r,\theta,\phi))+\frac{1}{r^2}\frac{1}{\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta\Phi(r,\theta,\phi))+\frac{1}{r^2}\sin(\theta)\partial_\phi^2\Phi(r,\theta,\phi)=-\frac{1}{\epsilon_0}\rho(r,\theta,\phi) \tag{38}$$

Assume $\Phi(r,\theta,\phi)=F(r)G(\theta)H(\phi)$

$$\frac{\nabla^2(F(r)G(\theta)H(\phi)=(\partial_\phi^2H)(G(\theta)}{\sin^2(\theta))(\frac{F(r)}{r^2})+\frac{1}{sin(\theta)}\partial_\theta sin(\theta)\partial_\theta G(\theta)H(\phi)+\frac{1}{r^2}\partial_r r^2\partial_r F(r)H(\theta)G(\phi)} \tag{39}$$

If we take

$$(\partial_\phi^2H)=-m^2H \tag{40}$$

$$\frac{1}{sin(\theta)}\partial_\theta sin(\theta)\partial_\theta G(\theta)=(-l(l+1)+\frac{m^2}{\sin(\theta)^2})G(\theta) \tag{41}$$

$$\frac{1}{r^2}\partial_r(r^2\partial_rF(r)-\frac{l(l+1)}{r^2}F(r)=-\frac{1}{\epsilon_0}\frac{\rho(r,\theta,\phi)}{G(\theta)H(\phi)} \tag{42}$$

If m integer, positive or negative, $\frac{H=e^{im\phi}}{\sqrt{2\pi}}$ If l integer and $l(l+1)>m^2$ to make solution $G(\theta)$ regular at $\theta=0$ and 2π then

$$G=P_lm(cos(\theta)) \tag{43}$$

the associated Legendre polynomial. Then

$$G(\theta)H(\phi)=Y_{lm}(\theta\phi) \tag{44}$$

the spherical Harmonics, normalized so that

$$\int_0^{2\pi} \int_0^{\pi} Y^*_{lm}(\theta,\phi) Y_{lm}(\theta,\phi) sin(\theta) d\theta d\phi = \delta_{l,l'}\delta_{m,m'} \tag{45}$$

we can expand ρ and Φ in terms of Y_{lm}

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F_{lm}(r) Y_{lm}(\theta,\phi) \tag{46}$$

$$\rho = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \rho_{lm}(r) Y_{lm}(\theta,\phi) \tag{47}$$

so

$$\frac{1}{r^2}\partial_r r^2 \partial_r F_{lm}(r) - \frac{l(l+1)}{r} F_{lm}(r) = -\frac{1}{\epsilon_0}\rho_{lm}(r) \tag{48}$$

$$\rho_{lm}(r) = \int \rho(r,\theta,\phi) Y^*_{lm}(\theta,\phi) sin(\theta) d\theta d\phi \tag{49}$$

Conductors:

A conductor is a medium in which there are electrons, or positive "holes" free to move. Since an electric field would push around the charges, the electric field inside a conductor (in the static limit) must be 0. Any charges lie on the surface of the conductor. Since curl E is zero, the E field just outside the conductor parallel to the surface must equal 0. This also means that the potential on the surface must be constant. The perpendicular component of E just outside the surface is equal to the charge density on the surface divided by ϵ_0.

A Type 1 superconductor is a material within which the B field equals 0. Since the divergence of B, in magnetostatics, is 0, the perpendicular component of B must also be 0 just outside the superconductor. The parallel component of must equal the μ_0 times the surface current density in the direction perpendicular to the magnetic field just outside the surface.

Image charges:

If one has a charge outside a conductor, the surface of the conductor is an equipotential. For symmetrical systems, one can often cancel, or set to 0, the potential on the surface, by imagining that there is another charge inside the conductor.

$$G_F(\vec{x},\vec{x}') = \frac{1}{4\pi\epsilon_0\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{4\pi\epsilon_0\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

is, (for example 0 at z=0, for all values of x, y, x', y' So if a charge is located at $x'.y', z' > 0$ then this potential will make $z = 0$ an equipotential, this would be the solution for the potential for a charge at x', y', z' over a conductor at $z = 0$. The image charge inside the conductor at $x', y', -z'$ does not really exist, except of the surface charge, except that implied by the surface electric field $E_z = \partial_z G(\vec{x}.\vec{y})$.

The two cases in which image charges work are for a plane surface and for a spherical conductor. In that second case, the image charge is stronger than the real charge, and is closer to the inside surface.

One can also solve more difficult cases – eg a right angled corner of a conductor, by allowing the image charges to also have image charges in the second plane conductor.

Green's theorem:

Given a potential in a space surrounded by a boundary. Given that the potential is known on that boundary. The the potential inside is determined by the potential values on the boundary.

Find a solution to the equation

$$\nabla^2 G_F(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}') = 0$$

With the boundary condition that if $\vec{x}$ is on the boundary, then $G(\vec{x},\vec{x}')$ is zero. Note that this is the same as if one imagined that the boundary a conductor, and $\vec{x}'$ is the location of a charge.

Then the potential at the point $\vec{x}'$ is equal to the integral over the boundary of the known potential at the boundary times the normal derivative on that point of the boundary of the function G.

$$\Phi(\vec{x}') = \int_{bndry} \Phi(\vec{x}) n^i \partial_i G_F(x,x') d^2\text{surface} \tag{50}$$

where n^i is the unit vector perpendicular to the boundary pointing into the region.

(There is an equivalent version if one specifies the perpendicular derivative of the potential on the boundary, but less often used.)

Aharonov-Bohm effect:

In region where $B = 0$, we can always find gauge Λ such that $\vec{A} = 0$.

$$\Lambda = -\int_{\vec{x}_0}^{\vec{x}} A_i(\vec{x}(\lambda))\frac{dx^i}{d\lambda}d\lambda \tag{51}$$

since Λ independent of path as long as the two paths do not encircle area where $\vec{B} \neq 0$, and there exists no surface with paths as boundary which intersects area where $\vec{B} \neq 0$. If there exist two paths whose surface does encircle such an area, then

$$\Delta\Lambda = \Lambda_(path1)(x) - \Lambda_{path2}(x) = \int B_\perp d^2\mathfrak{S} \tag{52}$$

Massive field

$$\partial_t^2 Q(t,l) + ((\partial_l - i\frac{e}{\hbar A_l}^2)Q(t,l) + M^2 Q = 0 \tag{53}$$

where Q is the massive charged field, $\frac{e}{\hbar}$ is constant that converts units of A to units of inverse length (it is another constant of nature– it does not necessarily have anything to do with quantum), and M is mas of field. Solution

$$G \approx e^{-i\omega t + kl + \int \frac{e}{\hbar}A_l dl} = e^{-i(\omega t+k)l}e^ie/\hbar\Lambda(x) \tag{54}$$

where $k = \sqrt{\omega^2 - M^2}$. If the particle goes along two paths with the same lengths, then the phases of the field along the two paths is determined by the difference in Λ, which is the determined by the magnitude of the B field the two path enclose. Ie, the A field has measurable properties even if the particle never sees a B field.

Adiabatic work by lowering two Dipoles:

For electric dipoles, the interaction energy in the EM field is directly related to the work done in lowering the dipoles toward each other. For magnetic dipoles, the lowered dipole cannot transfer energy to or from the EM field. The top lowered dipole simply transfers energy from internal mechanical energy of the dipole to the work done in lowering the dipole. There is however energy transferred to the EM field by the other stationary dipole, because of the electrical interaction (due to the changing B field of the other dipole producing an electrical field which transfers energy by $\vec{J}\cdot\vec{E}$ from the mechanical energy of the lower dipole out the EM field.

EM fields and matter

Electric:

$$\vec{P}_\alpha = \int (\vec{x}' - \vec{x}_\alpha)\rho_\alpha(x')d^3\mathfrak{V}'_\alpha \tag{55}$$

$$< P > (x) = \sum_\alpha f(\vec{x} - \vec{x}_\alpha)P_\alpha \tag{56}$$

where αlabels unit (atom, molecule,..) with self contained charge density. Assume total charge of unit is 0.

$$\vec{D}(x) = \epsilon_0 < \vec{E}(\vec{x}) >< \vec{P} > (\vec{x}) \tag{57}$$

$$\nabla\times < \vec{E} > (x) = 0; \quad \nabla\cdot\vec{D} = \rho_f(x) \tag{58}$$

Magnetic:

$$\vec{M}_\alpha = \frac{1}{2}\int (\vec{x}' - \vec{\alpha}) \times \vec{J}_\alpha d^3\mathfrak{V}' \tag{59}$$

and

$$< \vec{M} > (x) = \sum_\alpha f(\vec{x} - \vec{x}_\alpha)\vec{M}_\alpha \tag{60}$$

$$\vec{B} = \mu_0(\vec{H}(x) + < \vec{M} > (x)) \tag{61}$$

$$\nabla\times\vec{H} =< \vec{J}_f > (x); \quad \nabla\cdot\vec{B} = 0 \tag{62}$$

If ρ_f and $\vec{J}_f$ are zero, Boundary conditions where $\vec{M}$discontinuous

$$H_\parallel \text{ continuous}; B_\perp \text{ continuous} \tag{63}$$
$$E_\parallel \text{ continuous}; D_\perp \text{ continuous} \tag{64}$$

If$\vec{P}$ and $\vec{M}$ linear functions of H and E, then

$$\vec{D} = \epsilon\vec{E} \tag{65}$$
$$\vec{B} = \mu\vec{E} \tag{66}$$

At low frequencies, $\epsilon > \epsilon_0$, and usually $\mu > \mu_0$

If $\vec{M} \neq 0$ even when $\vec{H} = 0$, permanent magnet (ferromagnetic).
