Physics 530-21
Assignment 2

1.[3= half per subquestion] Assume that $H^A{}_B$, $L_A{}^B{}_C$ and M_{AB} are tensors, and $f,\ g$ are functions. Which of the following are tensors and why?
i)$Q_i{}^j = H^l{}_i$

Yes. Left is function of tangent vector assoc with A and cotangent with B, while rhs has tangent associated with A and cotang with B.
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ii)$R = H^i{}_i$
——————————————— Yes, contraction on right to give scalar while left is a scalar.
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iii)$T_{ijk}{}^l = H^l{}_i M_{jk}$

Yes, the transformations on both sides are the same/
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iv)$T_{ijk}{}^l = H^l{}_i + M_{jk}$

No. Not linear on right. If one looks at the change in coordinates, we would have

$$\tilde{T}^p_{mno}\partial_i\tilde{x}^m\partial_j\tilde{x}^n\partial_k\tilde{x}^o\partial_{\tilde{p}}x^l = T_{ijk}{}^l$$
$$= H^l{}_i + M_{jk} = \tilde{H}^p_m\partial_{\tilde{p}}x^l\partial_i\tilde{x}^m + \tilde{M}_{no}\partial_j\tilde{x}^n\partial_k\tilde{x}^o$$

Clearly the far left and far right cannot equal each other. In a simpler term, if $A = B + C$ then it is not true that $(Aef = Be + Cf)$ for arbitrary e abd f.
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v) $R^i = L_j{}^i{}_j$

No. Different arguments on left and right. On right B is not a contraction. A sum over two cotangent components is not coordinate invariant. The summation convetion is ONLY true for one tangent and one cotangent component.
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vi) $S_i = L_i{}^j{}_j - L_j{}^j{}_i$

Yes, On left we have a single cotangent vector, and the repeated index is one tangent and one cotangent index summed over. In that sum coordinate transformation of the two repeated indices are inverses of each other, and cancel in the sum.. The j indices indicate contraction, some tangent or cotangent function. .
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2. [7.5– 1/2 for components of each, 1/2 for each length] Given coordinates r, θ, what are the tangent vectors to the curves defined by the coordinate

conditions

$$r(\lambda) = r_0 \tag{1}$$
$$\theta(\lambda) = \lambda \tag{2}$$

$$T^r = \frac{dr}{d\lambda} = 0 \tag{3}$$
$$T^\theta = \frac{d\theta}{d\lambda} = 1 \tag{4}$$

are the two components of the tangent vector to that curve.

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$$r(\lambda) = \lambda \tag{5}$$
$$\theta(\lambda) = 5 * \lambda \tag{6}$$

$$T^r = \frac{dr}{d\lambda} = 1 \tag{7}$$
$$T^\theta = \frac{d\theta}{d\lambda} = 5 \tag{8}$$

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$$r(\lambda) = 10\lambda \tag{9}$$
$$\theta(\lambda) = 50 * \lambda \tag{10}$$

$$T^r = \frac{dr(\lambda)}{d\lambda} = 10 \tag{11}$$
$$T^\theta = \frac{d\theta}{d\lambda} = 50 \tag{12}$$

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What is the cotangent vector of the following functions

$$f(r, \theta) = r^2 \tag{13}$$

$$W_r = \partial_r f(r, \theta) = 2r W_\theta = \partial_\theta f(r, \theta) = 0 \tag{14}$$

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$$f(r,\theta) = r^2 + \theta^2 \tag{15}$$

$$W_r = \partial_r f(r,\theta) = 2r \tag{16}$$
$$W_\theta = \partial_\theta f(r,\theta) = 2\theta \tag{17}$$

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In each case find the lengths of these various vectors for each point at which they are defined if the metric is given by a)

$$ds^2 = dr^2 + d\theta^2 \tag{18}$$

$$g_{rr} = 1 \quad g_{r\theta} = 0 \quad g_{\theta r} = 0 \quad g_{\theta\theta} = 1 \tag{19}$$
$$g^{rr} = 1 \quad g^{r\theta} = 0 \quad g^{\theta r} = 0 \quad g^{\theta\theta} = 1 \tag{20}$$

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$$ds^2 = dr^2 + r^2 d\theta^2 \tag{21}$$

$$g_{rr} = 1 \quad g_{r\theta} = 0 \quad g_{\theta r} = 0 \quad g_{\theta\theta} = r^2 \tag{22}$$
$$g^{rr} = 1 \quad g^{r\theta} = 0 \quad g^{\theta r} = 0 \quad g^{\theta\theta} = \frac{1}{r^2} \tag{23}$$

Thus the lengthsquared of the three tangent vectors are
first metric

$$i) \quad g_{ij}T^iT^j = g_{rr}T^rT^r + g_{r\theta}T^rT^\theta + g_{\theta r}T^\theta T^r + g_{\theta\theta}T^\theta T^\theta \tag{24}$$
$$= 1 \tag{25}$$
$$ii) 26 \qquad iii) 2600 \tag{26}$$

The second:

$$i)\frac{1}{r^2} = \frac{1}{r_0^2} \quad ii) 1 + \frac{25}{r^2} = 1 + \frac{25}{\lambda^2} \tag{27}$$
$$iii) 100 + \frac{2500}{r^2} = 100 + \frac{25}{\lambda^2} \tag{28}$$

For the cotangent vectors we have

first metric length squared:

$$i) g^{ij} W_i W_j = g^{rr} W_r W_r + g^{r\theta} W_r W_\theta + g^{\theta r} W_\theta W_r + g^{\theta\theta} W_\theta W_\theta = 4r^2 \tag{29}$$

$$ii) 4r^2 \tag{30}$$

second metric

$$i) 4r^2 \quad ii)\ 4r^2 + \frac{4\theta^2}{r^2} \tag{31}$$

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3. Consider the two sets of coordinates x, y and r, θ where

$$r(x, y) = +\sqrt{x^2 + y^2} \tag{32}$$

$$\tan(\theta) = \frac{y}{x} \tag{33}$$

[2]What are x and y in terms of r and θ?

These are just the usual 2-D polar coordiantes.

$$x = r\cos(\theta) \quad y = r\sin(\theta) \tag{34}$$

$$y = x\tan(\theta) \to r = (\sqrt{x^2(1 + \tan(\theta)^2)} = \frac{x}{cos(\theta)} \tag{35}$$

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[2]If we define x, y as x^1, x^2 and r, θ as $\tilde{x}^1, \tilde{x}^2$, what are the two Jacobian matrices

$$\partial_j \tilde{x}^i \text{ and } \partial_{\tilde{j}} x^i \tag{36}$$

$$\partial_r x(r, \theta) = cos(\theta) \quad \partial_r y(r) = sin(\theta) \tag{37}$$

$$\partial_\theta x(r, \theta) = -rsin(\theta) \quad \partial_\theta y = rcos(\theta) \tag{38}$$

$$\partial_x r = \frac{x}{\sqrt{x^2 + y^2}} = \cos(\theta)$$

$$\partial_x \theta = \partial_x arctan(\frac{y}{x}) = \frac{-y}{x^2 + y^2} = -\frac{sin(\theta)}{r}$$

$$\partial_y r = \frac{y}{\sqrt{x^2 + y^2}} = \frac{sin(\theta)}{r}$$

$$\partial_y arctan(\frac{y}{x} = \frac{x}{x^2 + y^2} = \frac{cos(\theta)}{r}$$

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[3]If the metric for x, y is

$$ds^2 = dx^2 + dy^2 \tag{39}$$

What is the metric in terms of $r, \theta, dr, d\theta$?

$$\begin{aligned}
dx &= d(r\cos(\theta)) = drcos(\theta) + r(-sin(\theta)d\theta) \\
dy &= (d(r\sin(\theta) = drsin(\theta) + r(cos(\theta)d\theta \\
dx^2 + dy^2 &= (drcos(\theta) + r(-sin(\theta)d\theta)^2 + (drsin(\theta) + r(cos(\theta)d\theta)^2 \\
&= dr^2 + r^2 d\theta^2
\end{aligned}$$

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4. Given that the metric for x^1, x^2, x^3 is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + 4(dx^3)^2 \tag{40}$$

[1] what are the components of the metric g_{ij}?

[2] What are the components of g^{ij} and what is $\sqrt{g}$? ————————————————-

$$g_{11} = 1; \qquad g_{22} = 1; \qquad g_{33} = 4 \tag{41}$$

$$g_{12} = g_{21} = g_{23} = g_{32} = g + 13 = g_{31} = 0 \tag{42}$$

This is a matrix of the form

$$Matrix(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{43}$$

and the determinant of this matrix is 4, and the square root of that determinant is 2.

The inverse components are

$$g_{11} = 1; \qquad g_{22} = 1; \qquad g_{33} = \frac{1}{4} \tag{44}$$

$$g_{12} = g_{21} = g_{23} = g_{32} = g_{13} = g_{31} = 0 \tag{45}$$

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5. In cylindrical coordinates (r, θ, z), the metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{46}$$

[3] Consider the vector potential $A_r = cos(\theta), \quad A_\theta = r\sin(\theta), \ A_z = 1$ Find the components of $B^i = \epsilon^{ijk}\partial_j A_k$

[1] What are the components of B^i if we change the sign of A_θ in the above?

We take $x^1 \equiv r, \quad x^2 \equiv \theta, \quad x^3 \equiv z$ just to remind ourselves what the names of the three coordinates are. Note giving them a name like r does not necessarily mean anything physically. It is just a name. A name does not have a personality, it is just an arbitrary label. However in this case looking at the metric suggests that these names are special, r being the distance from the center to the point of interest, θ the angle around that center line, and z the distance perpendicular to the r and θ lines.

$$g_{rr} = 1; \quad g_{\theta\theta} = r^2 \quad g_{zz} = 1 \tag{47}$$

$$\text{Off diagonal terms are all!0} \tag{48}$$

$$\sqrt{g} = r \tag{49}$$

$$\epsilon^{123} \equiv \epsilon^{r\theta z} = \frac{1}{\sqrt{g}} = \frac{1}{r} \tag{50}$$

$$B^i = \epsilon^{ijk}\partial_j A_k \tag{51}$$

$$B^r = \epsilon^{rjk}\partial_j A_k = \epsilon^{r\theta z}\partial_\theta A_z + \epsilon^{rz\theta}\partial_z A_\theta \tag{52}$$

$$= \frac{1}{\sqrt{g}}(\partial_\theta A_z - \partial_z A_\theta) \tag{53}$$

$$= \frac{1}{r}(0 - 0) \tag{54}$$

$$B^\theta = \frac{1}{r}(\partial_z A_r - \partial_r A_z) = 0 \tag{55}$$

$$B^z = \frac{1}{r}(\partial_r A_\theta - \partial_\theta A_r) = \frac{1}{r}(sin(\theta) - (-sin(\theta)) = \frac{2}{r}sin(\theta)) \tag{56}$$

If we change the sign of A_θ then only the last term changes, and we get

$$B^z = \frac{1}{r}(\partial_r A_\theta - \partial_\theta A_r) = \frac{1}{r}\frac{1}{r}(-sin(\theta) - (-sin(\theta)) = 0.$$

Ie, with the changed sign of A_θ the non-zero vector potential gives 0 magnetic field. Ie, there are non-zero functions of A_i which give 0 magnetic field. These are called different gauges for the same B fields.

(Note that with the closure of the University on Wed, we will not have covered what the relation is between A and B, and thus this will be a bonus problem for those who actually read the notes.)

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