Physics 301-24

Assignment 3 [24]+[5]bonus

1.i)[3] Show that the solution for the potential of a spherical layer of charge at $r = R$ with total charge Q is given by

$$\phi(r,\theta,\phi) = \begin{cases} \frac{Q}{4\pi r} & r > R \\ \frac{Q}{4\pi R} & r < R \end{cases} \tag{1}$$

Use spherical coordinates.

This is a spherically symmetric problem, so let us go to spherical coordinates. We know shat the potential is outside the sphere. Inside, the two solutions go as a constant, or as $1/r$. Since the $1/r$ terms will give a charge at the center, it must be a constant there. Furthermore at the surface if the potential is not continuous, there would be a step function at r, and the first derivative would be a delta function, and the second would be a deriative of a delta function, which even when one integrates it with respect to one would get 0. Ie, it must be continuous. Outside the $r = R$, the potential is $Q/4\pi r$, inside it must be $Q/4\pi R$, a constant.

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ii)[2] The alternative would be to do the integral using the Green's function in Cartesian coordinates. Write down the integral you would have to do in order to get the potential everywhere. The integral is non-trivial and I do not believe you can do it.

The answer is to write the Green's funtion

$$\phi(x,y,z) = \int \frac{1}{\epsilon_0}\rho(x')G(x,y,z;x',y',z')\rho(x',y',z')dx'dy'dz' \tag{2}$$

$$\tag{3}$$

where

$$\rho(x',y',z') = \frac{Q}{4\pi\epsilon_0\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\delta(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}-R) \tag{4}$$

In order to do the integral, one first has to figure out what the integral over the delta function is.

$$\int g(x)\delta(f(x))dx = \int (\delta(f(x))\frac{1}{\partial_x f(x)}d(f(x)) = g(x_0)\frac{1}{\partial_x f(x_0)} \tag{5}$$

where $f(x_0) = 0$. As you can see this is getting to be very messy.

However, if we transform to polar coordinates define

$$z = rcos(theta); \quad x = rsin(theta)cos(\phi) \quad y = rsim(\theta)son(\phi) \tag{6}$$

and similarly for x', y', x' so

$$\rho = \int (\frac{Q}{4\pi r'^2}\delta(r'-R)) \tag{7}$$

$$\tag{8}$$

$$\Phi(x) \tag{9}$$

$$= \frac{\sigma}{4\pi\epsilon_0\sqrt{r^2+r'^2-2(xx'+yy'+zz')}}\delta(r')r'^2 sin(\theta)dr'd\theta'd\phi'$$

$$= \frac{\sigma}{4\pi\epsilon_0}\frac{1}{4\pi\epsilon_0\sqrt{r^2+R^2-2rR(cos(\theta)\cos(\theta')+sin(\theta)sin(\theta')(cos(\phi-\phi'))}}d\cos(\theta')d\phi'$$

Again this integral is difficult to evaluate.

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iii)[2] What is the Electric field inside $r = R$. (Note that in the gravitational force case, Newton had to go through much argument to come up with this result in the gravitational case.)

For a spherically symmetric system, the two possible solutions are C/r and D. The latter is a constant. The former has a δ function source at r=0. The former is a possible solution outside $r = R$ but not inside, as it would produce and extra charge at the center, which is not part of the specification of the problem. The latter however is possible.

Thus, inside the shell, the potential is a constant, and the gradient of a constant is zero. Thus all compoents of the field inside are 0

Note also as mentioned in class, the potential must be continous as otheresise the gradiant of the potential will have the derivative of a step function, whic is a delta function which means that the integral of the energy, which as I said goes as $\vec{E}\cdot\vec{E}$, which will be a a delta function squared, whose integral is infinite. One cannot have an infinite energy, so the potential must be everywhere continuous.

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iv)[1] Translate the above to cartesian x, y, z coordinates, assuming that the origin of the spherical polar coordinates is at $x = 0,\ y = 0, z = 0$.

The translation is just that $r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

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2.i)[3] Assume we have two point charges with opposite $\pm Q$, the positive one centered at $x = a,\ y = 0,\ z = 0$, and the other at $x = -a,\ y = 0,\ z = 0$, Using Gauss's thm., or any other argument, what would the integral over a spherical circle of radius $\mathcal{R} >> \mathcal{R}, \dashv$ centered at $x = 0,\ y = 0,\ z = 0$.

Each charge has a potential of $Q/4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})$ where x', y', z' is the location of the particle. The answer the two charges is then the sum of the answers to the individual one. Thus

$$\phi = -\frac{1}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+(y)^2+(z)^2}} - \frac{1}{\sqrt{(x+a)^2+(y)^2+(z)^2}}\right) \tag{10}$$

The integral over a surface encompassing the two charges would have a total charge of 0

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ii)[4] What is the electric field along the z axis ($x = y = 0$) and along the x axis (y=z=0)?

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E_z for $x = y = 0$

$$E_z = -\partial_z\phi = -\partial_z\frac{Q}{4\pi\epsilon_0}(\frac{1}{\sqrt{(-a)^2+z^2}} - \frac{1}{\sqrt{(a)^2+z^2}}) = -\partial_x(0) = 0. \tag{11}$$

E_x for $y = z = 0$

$$E_x = -\partial_x\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|x-a|} - \frac{1}{|x+a|}\right) = \frac{Q}{4\pi\epsilon_0}\left(\frac{\partial_x|x-a|}{|x-a|^2} - \frac{\partial_x|x+a|}{|x+a|^2}\right) \tag{12}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{\Theta(x-a)-\Theta(a-x)}{|x-a|^2} - \frac{\Theta(x+a)-\Theta(a+x)}{|x+a|^2}\right) \tag{13}$$

For $|x| > a$, this becomes

$$\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right) = \frac{Q}{4\pi\epsilon_0}\left(4\frac{a|x|}{(x^2-a^2)^2}\right) \tag{14}$$

both above and below the charges, the E field falls off as $1/x^3$ and points in the same direction. Between the charges it points in the opposite direction.

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iii)[4] Far away from the charges ($r >> a$) what is the potential to lowest non-zero order in a/r?

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We have the terms

$$\begin{aligned}
&\frac{1}{\sqrt{(x-a)^2+y^2+z^2)}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2)}} \\
&\quad = \frac{(\sqrt{(x+a)^2+y^2+z^2)} - \sqrt{(x-a)^2+y^2+z^2)})}{\sqrt{(x-a)^2+y^2+z^2)}\sqrt{(x+a)^2+y^2+z^2)}} \\
&= \frac{\sqrt{r^2+a^2-2ax} - \sqrt{r^2+a^2+2ax}}{\sqrt{(r^2+a^2)^2-16a^2x^2)}} \\
&\approx \sqrt{r^2+a^2} - 2ax/\frac{\sqrt{r^2+a^2}}{r^2+a^2-2a^2x^2/(r^2+a^2)} \\
&\approx -\frac{2ax}{(r^3)}.
\end{aligned} \tag{15}$$

This is a dipole electric potential.

3. i)[5] Using the Green's function $G(\mathfrak{X}, \mathfrak{X}') = -\frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$ find the potential for a line of charge, with charge density per unit length of

$\rho(x,y,z) = \sigma\delta(x)\delta(y)$. Compare this with the potential you derived in Assignment 2 for the potential for a cylindrically symmetric charge distribution.

The easiest way to do this is to go to cylindrical polar coordinates.

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 \tag{16}$$

Now, the charge distribution is symmetric under rotations around the z axis, and so should the solution be. Thus the answer should be independent of θ. Also the distribution is independent of z transformations, so the solution should be indepedent of z. Thus we have

$$\frac{1}{\rho}\partial_\rho \rho \partial_\rho \phi(\rho) = 0 \tag{17}$$

or $\phi(\rho) = C\ln(\rho) + D$. The term D is a constant whose graident is 0, so we can eliminate it. Then the only compoent of E is the ρ component.

Instead of a sphere, create a cylinder of unit height, and radius R. On the top and bottom of the cylinder, the E field is parallel to the surface. on the sides, E is in the ρ direction which is perpendicular to the surface. Now, $g_{\rho\rho} = g^{\rho\rho} = 1$, so $E^\rho = E_\rho$, and the integral of E^ρ over the surface is

$$\int E^\rho R d\theta dz = E_\rho 2\pi R L = Q/epsilon_0 = \sigma L \epsilon_0. \tag{18}$$

since $\phi = C\ln(\rho)$, $E_\rho = -\frac{C}{\rho}$ which implies that $\frac{C=-\sigma}{\epsilon_0}$.

The 3 dimensional Green's function is

$$G(x,y,z;x',y,'z') = -\frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{19}$$

so the solution is

$$\Phi(x,y,z) = -\int \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\frac{\sigma}{\epsilon_0}\delta(x')\delta(y')dx'dy'dz' \tag{20}$$

$$= -\int \frac{1}{4\pi\sqrt{(x)^2+(y)^2+(z-z')^2}}\frac{\sigma}{\epsilon_0}dz' \tag{21}$$

$$\tag{22}$$

The problem is that that integral is infinite. Whether that is just an infinite constant, or is a function of the coordinates depends on how you approach infinity. If one assumes that the solution should be symmetric around z=0 say, and has a z tranlation symmetry, one should get the same result as with the cylindrical coordinates.

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ii) [5 bonus] Now let us assume you have a line of charge, but with endpoints at $z = \pm a$ie $\rho(x,y,z) = \sigma\delta(x)\delta(y)\Theta(z+a)\Theta(a-z)$. Find the potential for this distribution.

(Recall that $\Theta(z)$ is the step function, such that it is 1 for $z > 0$ and 0 for $z < 0$.)

a) What is the derivative with respect to z of the above charge distribution?

The derivative of the Θ function is a delta function. The derivative of of $\Theta(-z)$ is minus the delta function. (chain rule.)Thus the end points have two charge distribution of $\delta(z+a)$ and $-\delta(z-a)$. Ie two point charges with charge σ at $z=-a$ and $-\sigma$ at $z=a$. Ie, this is just the problem of question 2.

Now,

$$\int G_\rho(x,y,z;x'y'z')\partial_z\rho(x',y',z')dx'dy'dz'$$

$$= -\int \partial'_z G_\rho(x,y,z;x'y'z')\rho(x',y',z')dx'dy'dz' \tag{23}$$

$$= \partial_z \int G_\rho(x,y,z;x'y'z')\rho(x',y',z')dx'dy'dz' \tag{24}$$

$$= \partial_z \Phi(x,y,z) \tag{25}$$

The first transformation is by an integration by parts (the boundary end points are because ρ is zero at infinity), and the transfer to the z derivative is because G_ρ is a function of $z-z'$. Thus the z derivative of Φ is just the potential of two oppositely signed charges at and -a with charges $-\sigma$ and σ. We thus have

$$\Phi = \int_{-\infty}^{z} \frac{\sigma}{4\pi\epsilon_0}\left(-\frac{1}{\sqrt{(z'-a)^2+x^2+y^2}} + \frac{1}{\sqrt{(z'+a)^2+x^2+y^2}}\right)dz' \tag{26}$$

$$= -\int_{z-a}^{z+a} \frac{1}{\sqrt{z"^2+x^2+y^2}}dz" \tag{27}$$

for large values of z this is approximately

$$\Phi \approx -\frac{2a\sigma}{4\pi\epsilon_0\sqrt{z"^2+x^2+y^2}} \tag{28}$$

to lowest non-zero order in a and for large $x^2+y^2+z^2$. $2a\sigma$ is the total charge in the strip of charge.

We note that the integrand in evaluating Φ is antisymmetric in z and thus the integral is symmetric.

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b)Find the potential for charge distribution $\frac{d\rho(x,y,z)}{dz}$. Argue that this is the z derivative of the potential for the distribution in 3.i). Find the potential for the charge distribution in 3.i in the limit as z gets very large ($|z| >> a$)

c) Alternatively you can use the 3 dimensional Green's function to calculate the potential. Not entirely surprizingly you find that either way the integrals you need to carry out is similar.

Hint. The integral

$$\int_0^M \frac{1}{\sqrt{mu^2+\nu^2}}d\mu = -\ln(|\nu|) + \ln(M+\sqrt{M^2+\nu^2}) \tag{29}$$

This actually a red herring, since as we saw above we do not need it., although it does give the full integral

$$\int_{z-a}^{z+a} \frac{dz}{\sqrt{z'^2+x^2+y^2}} dz' \tag{30}$$

$$= \int_0^{z-a} \frac{dz}{\sqrt{z'^2+x^2+y^2}} - \int_0^{z-a} \frac{dz}{\sqrt{z'^2+x^2+y^2}}$$

$$= ln(z-a+\sqrt{(z-a)^2+x^2+y^2}) - ln(z+a+\sqrt{(z+a)^2+x^2+y^2}) \tag{31}$$

However this expression is not terribly helpful, and must be treated with care. I wasted about full days of work on this expression, chasing many mistakes in trying to use it.

Oh well. I set the problem thinging it would be relatively straightforward to solve, and found myself floudering until I finally got the simple precedure above.

I never thought I would find myself trapped in the old joke: A prof writes and equation on the board, and says– getting this is trivial. Suddenly he stops, stares the blackboard deep in thought, wanders out of the room, and finally comes back just before the class is to end, and says "Yes, it is trivial".

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