Physics 301-24
Assignment 4

[21+2bonus]

1. [5] Consider a point in space containing an electric potential. Consider a sphere around that point of radius R and within that sphere the potential is source free. Show that the potential at the center of the sphere is the average of the potential over the surface of the sphere. (Hint define a Green's function $\tilde{G}(\mathfrak{X},\mathfrak{X}')$ where $\mathfrak{X}'$ is the center of the sphere and $\mathfrak{X}$ is on the surface of the sphere. Find $\tilde{G}$ such that it is zero on the surface of the sphere. Use that to find the value of the potential at the center of the sphere in terms of the value of the potential on the surface.)

Green's thm says that the potential field inside a boundary equals the integral over the boundary of the normal derivative of the Green's funciton which is zero on the boundary and has the other point lying on the the point of interest, times the value of the potential on the boundary.

In our case the point of interest is the center of the sphere. The usual Greens function is the solution to

$$\nabla^2 G(\mathfrak{X},\mathfrak{X}') = \delta(\mathfrak{X},\mathfrak{X}') \tag{1}$$

$$G(\mathfrak{X},\mathfrak{X}') = -\frac{1}{4\pi\sqrt{(x-x')^2 + y-y')^2 + (z-z')^2}} \tag{2}$$

To make this into a Green's function which is zero on the boundary (which in this case is the sphere

$$(x-x')^2 + (y-y')^2 + (z-z')^2 = R^2 \tag{3}$$

is to add to this a solution of the source free equation which makes this zero on the boundary. But on the boundary $G(\mathfrak{X}, mfx')$ is a constant, namely $\frac{-1}{4\pi R}$. And a solution of the free equation is a consant, so,

$$\tilde{G}(\mathfrak{X}.\mathfrak{X}') = -\frac{1}{4\pi\sqrt{(x-x')^2 + y-y')^2 + (z-z')^2}} + \frac{1}{4\pi R} \tag{4}$$

The value of the field at the center, $\mathfrak{X}'$ is

$$\phi(\mathfrak{X}') = \int_{\mathfrak{S}} \partial_\perp \tilde{G}(\mathfrak{X},\mathfrak{X}')\phi(\mathfrak{X})d\mathfrak{S} \tag{5}$$

$$= \int_{r=R} \frac{1}{4\pi r^2}\phi(r,\theta\phi)r^2 sin(\theta)d\theta d\phi = \int\int \phi(R,\theta.\phi)\frac{sin(\theta)d\theta d\phi}{4\pi} \tag{6}$$

which says that $\phi(\mathfrak{X}')$ is just equal to the mean value of the field over the sphere at distance R away from the point $\mathfrak{X}'$. Note that this is independent of R.

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2)[5] show that the monopole ($l = 0$) moment of a charge distribution is just the total charge, and the components of the dipole moment are given by

$$\int \rho(x,y,z)(x_i y)dx^3; \quad \int \rho(x,y,z) z d^3x; \quad \int \rho(x,y,z)(x-iy)d^3x \tag{7}$$

The monopole component is just the $Y_{00} = \frac{1}{\sqrt{4\pi}}$ componment, and thus the potential is

$$\Phi_{00}(r,\theta,\phi) = -\frac{1}{\epsilon_0}\int \Theta(r-r')\frac{1}{r} + \Theta(r'-r)\frac{1}{r'}\frac{1}{4\pi}\rho(r',\theta',\phi')r'^2 sin(\theta')dr'd\theta'd\phi' \tag{8}$$

' Going far from the distribution of charge (r larger than the charge distribution size) we get

$$\Phi_{00}(r,\theta,\phi) = -\frac{1}{4\pi r}(\int \rho(r',\theta',\phi')r'^2\sin(\theta')dr'd\theta'd\phi' = -Q/4\pi\epsilon_0 r \tag{9}$$

Now,

$$Y_{10} = \sqrt{\frac{3}{4\pi}}cos(\theta) = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \tag{10}$$

$$Y11 = \sqrt{\frac{3}{8\pi}}sin(\theta)e^{i\phi} = sqrt\frac{3}{8\pi}sin(\theta)(cos(\phi) + isin(\phi)) = sqrt\frac{3}{8\pi}\frac{(x+iy)}{r} \tag{11}$$

$$Y1,-1 = Y11^* = sqrt\frac{3}{8\pi}\frac{(x-iy)}{r} \tag{12}$$

Thus

$$\Phi_{10}(x,y,z) = -\frac{1}{\epsilon_0 r^2}\frac{r'}{3r^2}\rho(x',y',z')\frac{3}{4\pi}(\frac{z}{r}\frac{z'}{r'}\rho(x',y',z')dx'dy'dz' = -\frac{z}{\epsilon_0 r^3}\frac{1}{4\pi}\int z'\rho(x',y',z')dx'dy'dz' \tag{13}$$

$$\Phi_{11}(x,y,z) = \frac{x+iy}{8\pi r^3}\int \frac{(x'+iy')}{r'}\rho(x',y',z')dx'dy'dz' \tag{14}$$

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3)[4] Consider a charge distribution with both a monopole ($l = 0$) and a dipole ($l = 1$ moment to the potential. Show that by changing the origin around which you calculate the spherical expansion, you can set the dipole moment to zero, but only if the monopole moment is not zero.

From 2), the dipole moment is proportional to the three terms $\int \rho(x',y',z')x'd^3x'$, $\int \rho(x',y',z')y'd^3x'$, Now consider changing the origin of the polar coordinates to $\tilde{x} = x' - X$, $\tilde{y} = \int \rho(x',y',z')x'd^3x'y' - Y$. $tildez = z' - Z$ where X, Y, Z are constants. Then we get

$$\int \rho(\tilde{x},\tilde{y},\tilde{z})\tilde{x}d^3\tilde{x} = \int \tilde{\rho}(\tilde{x},\tilde{y},\tilde{z})\tilde{x}d^3\tilde{x} \tag{15}$$

$$= \int \rho(x',y',z')(x'-X)d^3x' = \int \rho(x',y',z')x'd^3x' - X\int \rho(x',y',z')d^3x' \tag{16}$$

$$= \int \rho(x',y',z')x'd^3x - XQ. \tag{17}$$

where Q is the total charge.Thus if we translate x by $X = \int \rho(x', y', z')x'd^3x/Q$ the new x component of the dipole moment will be 0. Exactly the same arguement applies to the y and z components. Thus we can always make the dipole moment to be 0 by a suitable translating if the charge Q (the monopole moment) is non-zero.

Note that the same is true of the other moments. If the dipole moment is the first non-zero dompoent, one can always drive the quadrapole moment to zero by a translaton of the coordinates.

4) [7 +2bonus] Consider the potential in cylindrical ccordinates with metric

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 \tag{18}$$

Write the Poisson equation of this metric in the "seperation of variables" form.

Solve the angular and the z equation.

In the case that the solution is independent of z solve the radial equaiton.

In the case that the solution is not independent of z, the solutions of the radial equations are modified Bessel functions. Any second order differential equation has two indepdenent solutions. What are the behaviour of the two solution near $r = 0$. What are the behaviour of the two solutions near $r = \infty$. Note that in order to get regular solutions both at $r$near zero and r near infinity, one must have charges in the space. (This is another manifestation of the theorem tha tthat the Poisson equation without sources has only one regular solution, the potential is constant everywhere.

Note again that the seperation of variables works only if there are symmetries of the equation. In this case, translation of z and rotation around the z axis.

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[1]

$$\frac{1}{r}\partial_r r \partial_r \Phi + \frac{1}{r^2}\partial_\phi^2 \Phi + \partial_z^2 \Phi = -\frac{1}{\epsilon_0}\rho(r, \phi, z) \tag{19}$$

Assume we can write $\Phi = F(r)E(\phi)H(z)$ Then

$$FEH(\frac{1}{rF(r)}\partial_r r \partial_r F(r) + \frac{1}{r^2}\frac{1}{E(\phi)}\partial_\phi^2 E(\phi) + \frac{1}{H(z)}\partial_z^2 H(z) = -\rho(r, \phi, z) \tag{20}$$

[2]

We assume that the terms which are z and ϕ dependent are constants

$$\frac{1}{E(\phi)}\partial_\phi^2 E(\phi) = -m^2 \tag{21}$$

$$\frac{1}{H(z)}\partial_z^2 H(z) = -k^2 \tag{22}$$

which have solutions

$$H(z) = e^{ikz} \tag{23}$$

$$E(z) = e^{im\phi} \tag{24}$$

In order that $E(\phi+2\pi) = E(\phi)$ we need m to be an integer. Then in order that $\int E_m(\phi)^* E_{m'}(\phi) d\phi = \delta_{m,m'}$ we need $E_m(\phi) = \frac{1}{\sqrt{2\pi}}$

[2]

And if we want $\int_{-\infty}^{\infty} H_k(z)^* H_{k'}(z) dz = \delta(k-k')$ we need

$$H_k(z) = \frac{1}{\sqrt{2\pi}} e^{ikz} \tag{25}$$

Thus we can write the equation as

$$\Phi_(r,\phi,z) = F_{km}(r) \frac{1}{2\pi} e^{im\phi} e^{ikz} \tag{26}$$

$$\frac{1}{r}\partial_r r \partial_r F_{km}(r) - \frac{m^2}{r^2} F_{km}(r) - k^2 F_{km}(r) = -\frac{1}{\epsilon_0}\rho_{km}(r) = \frac{-1}{\epsilon_0}\int\int \rho(r,\phi,z) H_k(z)^* E_m(\phi)^* dz d\phi \tag{27}$$

Now we can write this in terms of a r green's function, such that

$$\frac{1}{r}\partial_r r \partial_r G_{km}(r,r') - (\frac{m^2}{r^2} + k^2) G_{km}(r,r') = \delta(r-r') \tag{28}$$

[2]

Let us first look at the situation where the density is independent of z. Then the only important parts are $k^2 = 0$ and thus we have the equation

$$\frac{\frac{1}{r}\partial_r r \partial_r F_{0m}(r) - m^2}{r^2 . F_{0m}(r) = 0} \tag{29}$$

each term reduces the power of r by 2, so it would seem that a simple power of r would work. Take $F_{0m}(r) = \alpha_n r^n$ we get

$$(\alpha_n)(n^2 - m^2) r^{n-2} = 0 \tag{30}$$

which gives the two solutions which go as $r^{\pm m}$. The term which goes as $r^{|m|}$ goes to zero as $r \to 0$, while the term which goes as $r^{-|m|}$ diverges at r=0, but is well behaved at $r \to \infty$. Thus the solution needs to go as $\frac{1}{r^{|m|}}$ for $r > r'$ and go as $r^| m|$ as $r \to \infty$. It also has to be continuous as $r = r'$. Thus

$$G_{0m}(r,r') = C\left((\frac{r'}{r})^{|m|}\Theta(r-r') + (\frac{r}{r'})^{|m|}\Theta(r'-r) \right) \tag{31}$$

is continuous at $r = r'$, its first derivative is discontinuous at $r - r'$

$$r\partial_r G(r,r') = C|m| \left(-\frac{r'}{r})^{|m|}\Theta(r-r') + (\frac{r}{r'})^{|m|}\Theta(r'-r) \right) \tag{32}$$

which has a step discontinuity of size $-2C|m|$ at $r = r'$. The derivative of this would give a δ function of size $-2C|m|$ times $\frac{1}{r'}$ or $C = -\frac{r'}{2|m|}$

Thus

$$\frac{1}{r}\partial_r r \partial_r G(r,r') - \frac{m^2}{r^2} G(r,r') = \frac{1}{r'}\delta(r-r') \tag{33}$$

$$G(r,r') = -\frac{r'}{2|m|}\left((\frac{r'}{r})^{|m|}\Theta(r-r') + (\frac{r}{r'})^{|m|}\Theta(r'-r)\right) \tag{34}$$

will give the δ function.

[2-bonus]

However, the above is a bit premature, since $m=0$ is somewhat anomolous. r^0 and r^{-0} are the same solution, not two different solutions. Looking at the equation of $m=0$, we have

$$\frac{1}{r}\partial_r r \partial_r G(r,r') = 0 \tag{35}$$

(which is true for $r \neq r'$ has the two solutions *constant* and $ln(r)$. Near $r=0$, only the constant is regular. As $r \to infty$, the $ln(r)$ solution would seem to be divergent as well. However, the derivative, which is the electric field in the radial direction, falls as $1/r$ Ie, it is (sort of regular.) as $r \to \infty$

[0]

Now what about $k \neq 0$ The solutions of the Homogeneous equaitons give

$$\frac{1}{r}\partial_r r \partial_r G_{km}(r,r') - (\frac{m^2}{r^2} + k^2)G_{km}(r,r') = 0 \tag{36}$$

If we multiply r by k, we get

$$\frac{1}{kr}\partial_{kr}(kr)\partial_{kr} - (\frac{m^2}{(kr)^2} + 1)G(kr,kr') = 0. \tag{37}$$

Near kr=0, this equation has solutions $(kr)^{\pm|m|}$ for which only the $(kr)^|m|$ is well behaved. Near $r \to \infty$, the solution goes as $e^{\pm kr}$, for which only the minus sign is well behaved at infinity. Th full solution is called the modified Bessel function. $I_i|m|0(kr)$ is well behaved near 0, while $K_|m|(kr)$ decreases exponentially as r goes to infinity.
