Physics 301-24
Assignment 5

1)[6] Using the static approximation for the fields:
a) Show that in free space between two infinite plane conductors the E field is perpendicular to the conductors and is constant. What is the force per unit area and direction that the EM field exerts on itself. or on the plates.
b) Above a superconductor, the B field is parallel to the conductor. Assume one has two parallel slabs of superconductor, what is the pressure of the field on itself

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The stress tensor

$$\Theta_{ij} = \frac{1}{2}\left(\epsilon_0 E_i E_j - \frac{1}{2}E_k E^k \delta_{ij} + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}B_k B_k \delta_{ij}\right) \tag{1}$$

Let us say that the plate is the plane z=0. The electric field is perpendicular to the conductor. The force across the z=0 surface is thus the above with $j = z$ and thus the force across that surface per unit area is

$$F_z = \epsilon_0(E_z E_z - \frac{1}{2}((E_x E^x + E_y^y + E_z^z)\delta_{zz}) = \tag{2}$$

$$= \epsilon_0(E_z^2 - \frac{1}{2}E_z^2 1) = \epsilon_0 E_z^2 \tag{3}$$

The B field is parallel to the surface of the conductor, so lets say it is B_x. Then

$$F_z = \frac{1}{\mu_0}(B_z B_z - \frac{1}{2}((B_x B^x + B_y^y + B_z^z)\delta_{zz}) = \frac{1}{\mu_0}(0 - \frac{1}{2}B_x^2 1) \tag{4}$$

Thus the E field and the B field have opposite signs for the pressure on the plate. As we know the E field attracts the plate , and the thus the B field above of the superconductor repels the plate.
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2)[4] Consider a straight circular wire carrying a current I alon the conductor. Thewire has a conductivity σ which is the proportionality of the E field to the current.

$$\vec{J} = \sigma\vec{E} \tag{5}$$

Assuming that the magnetic field around the wire is $B = \mu_0\frac{I}{r}$ in the tangential direction obeying the right hand rule:(Thumb of right hand points in direction of $\vec{J}$, then half open fist fingers point in direction of $\vec{B}$). What is the energy flux into the wire from the electromagnetic field just outside the wire?

The B field around the wire assumed to be in the z direction, will be in the axial direction, and by Stoke's them, it will have an amplitude of

$$2\pi r B_{tang} = \mu_0 \int J_z dS = \mu_0 I \tag{6}$$

where I is the total current in the wire. But to drive that current we will need and E field in the z dirction of

$$E_z = \frac{1}{\sigma}\frac{I}{A} \tag{7}$$

where A is the area of the wire. Lets say that the wire has a radius of R. Thus

$$E_z = \frac{I}{\pi R^2 \sigma}. \tag{8}$$

The B field at the surface, and tangential to the surface around the wire is

$$B_{tang} = \mu_0 \frac{I}{2\pi R} \tag{9}$$

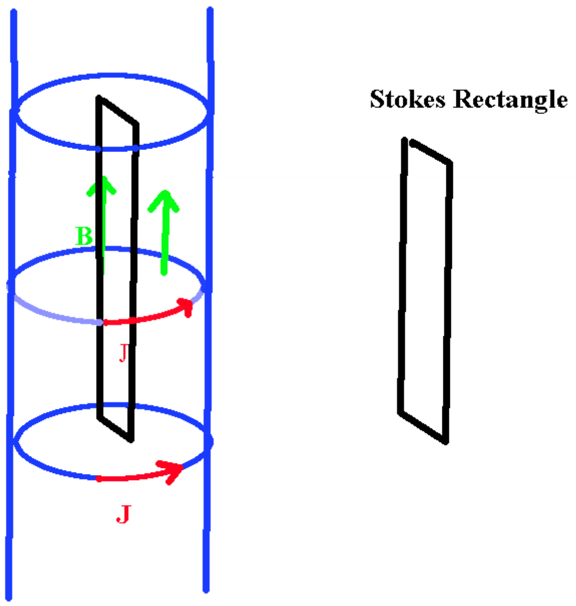

FIG. 1: Magnetic solinoid with Stokes rectangles

The Poynting vector which represents the energy flux is(since the E field parallel to the conductor is continuous, the E field just outside the wire must be the same as inside)

$$\epsilon_0\vec{E}\times\vec{B} = \frac{1}{\mu_0}E_zB_{tang} = -\frac{I^2}{2\pi^2R^3\sigma} \tag{10}$$

which is the radial energy flux per unit surface area of the conductor, so per unit length we get a flux of $\frac{I^2}{\pi R^2\sigma}$ into the wire. But $\frac{1}{piR^2\sigma}$ is just the resistance per unit length, so the energy flux into the wire is $I^2\mathcal{R}$ where $\mathcal{R}$ is the resistance per unit length of the wire.
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3)[5] Consider an infinite solenoid (a cylindrical hollow conductor) with a circumfrential uniform current flowing around the hollow. Assume by symmetry that the B field is directed in the direction of the axis of the cylinder. Argue that the B field is constant inside the cylinder and is zero outside the cylinder. Use Stokes theorem.

Use stoke's them with a rectangle oriented within the cylinder carrying the current. The integral of B parallel to the sides of the rectangle around the rectagle equals μ_0 times the current flowing through the surface of the rectangle. Since there is no current inside the cylinder, the surfce integral is 0. The B integral along the top and bottom of the rectangle is zero because the field is perpendicula to the sides at the top and bottom. Along the two vertical sides, the B integral is the B_1L along the one side and $-B_2L$ along theother side. But the line integral around the whole rectagle is 0. Thus $B_1 = B_2$ Since we can orient the rectagle anywhere inside, and can change the width of the rectangle, the B field must be everywhere the same. Outside the cylinder we can do the same which says that the B field everywhere outside is the same. all the way to infinity. Thus the outside energy in the B field would be infinite if B is not 0. The B field inside and outside is not the same since those rectangles go through the sides of the cylinder which has a current flowing through it, which would then lead to $B_1 \neq B_2$ and the B field inside would then equal the current per unit length times μ_0.
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4.)[6] Calculate the force between two electric ~~and two magnetic~~ dipoles if both are oriented so that the two dipole directions are parallel to each other and each is located along the axis of the other (the axis is the line running through one of the dipoles in the direction of the electric ~~or magnetic~~ dipole). You may assume that they are far apart from each other–ie, separated by much more than the diameter of the charge ~~or current~~ distributions. .

Note that you can model the dipole as two point charges, of opposite charge, separated by a small distance δ Keep only the lowest order terms in delta.

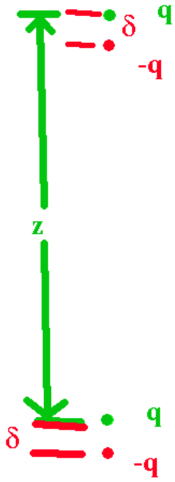

FIG. 2: electric dipole

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A much simpler way is to note that the dipole is essentially two opposite charges separated by a distance in the z direction. Thus the interaction energy would be to place two opposite charges separaed by δz along the z axis away from two oppoite charges a separated by delta z a distane z away as in the figure. . The force of one charge on the other at a distance r is $\frac{q_1q_2}{4\pi\epsilon_0 r^2}$ where the force is replusive if q_1q_2 positive. Thus the force of the top of the lower charges on the two on top isi proportional to $\frac{q^2}{z^2}-\frac{q^2}{(z-\delta)^2}$ due to the top charge of the bottom dipole, and $\frac{q^2}{z^2}-\frac{q^2}{(z+\delta)^2}$ due to the bottom. Thus the total force is

$$F=\frac{q^2}{4\pi\epsilon_0}(\frac{2}{z^2}-\frac{1}{(z+\delta)^2}-\frac{1}{(z-\delta)^2})=\frac{q^2}{4\pi\epsilon_0}\frac{2(z^2-\delta^2)^2-z^2((z-delta)^2+(z+\delta)^2)}{z^4(z^2-\delta^2)^2} \tag{11}$$

$$\approx-\frac{q^2\delta^2}{4\pi\epsilon_0}(\frac{6}{(z^4)}) \tag{12}$$

Now the dipole moment is $P=q(\delta/2)+(-q(-\delta/2))=q\delta$ so the force is attractive and equal to $6P^2/z^4$. If one modeled the magnetic dipole moment in the same way as made of two magnetic monopoles, one would get the same answer with $P\to M$ and $1/\epsilon_0\to\mu_0$. However the dipole moment is actually a function of the current, of finite size. The calculation is much more difficult, but the answer is the same as long as one stays far away from the dipole. If the monopoles are near each other, the answer differs. But then the higher multipole moments of the charge or the current distributions become important as well, and the argument becomes far more difficult.

In the loop magnetic case, the B_z component of the field gets weaker as you gt furthr away from the curent loop. But $\vec\nabla\cdot\vec B$ is zero (Maxwell's equation for the real EM field rather than the phoney "magnetic monopole" version.) sine $\partial_z B^z$ is non-zero, the B_x and B_y components must have derivatives just at the axis. By symmetry, thy must be pointing radialy awy from the axis. Thus near the axis the B field is radial, and then $\vec J\times\vec B$ must point in the z direction (where J is the curent of the otehr dipole, not producing the B field). This is what produces the force on the other dipole.

It is possible to calculate the force. One can calculate A^i using the Green's function, from the current, which we assume lies on a z=const sheet and flows around (tangentially to a constant x^2+y^2 circle) One finds that the A^i is zero along the z axis is linear in x and y around the z axis. Taking the curl, gives us B^i along the z axis. Using the $\nabla\cdot B=0$ the rate of change along the z axis and the circular symmetry of the problem give us both B_x and B_y in the vicinity of the z axis. Then integrating $\vec J\times\vec B$ for the second model dipole one finds the z component of the force caused by the two dipole models in the z direction. It turns out to be ideentical to the force between two electric dipoles, upon mapping M to P and $1/\epsilon_0$ to μ_0.

Again the behaviour if the two corrent loops are near each other deviates from that for the electric dipoles as one would expect, with again higher moments, the detailed structure of the current, causing deviations. While certainly not as simple to calculate due to all of the cross products, and due to the much more complex strucute of the magnetic dipole, it is both surprizing and telling that the results are so similar.
