Potential, delta function, and Green's function

We are going to be interested in the static limit of Maxwell equations, ie where the time derivative is negligible, and can be approximated by zero. This is clearly an approximation, since anything we do or measure has time dependence, but often it is a reasonable approximation.

Maxwell's equations then reduce to (Using Φ for the potential so as not to confuse it with the coordinate ϕ).

$$1 \quad E_i = -\partial_i \Phi \tag{1}$$

$$1' \quad \epsilon^{ijk}\partial_j E_j = 0 \tag{2}$$

$$2 \quad B^i = \epsilon^{ijk}\partial_j A_k \tag{3}$$

$$2' \quad \frac{1}{\sqrt{g}}\partial_i \sqrt{g} B^i = 0 \tag{4}$$

$$3 \quad \frac{1}{\sqrt{g}}\partial_i \sqrt{g} g^{ij} E_j = \frac{1}{\epsilon_0}\rho(t, \mathfrak{X}) \tag{5}$$

$$4 \quad \epsilon^{ijk}\partial_j g_{kl} B^l = \mu_0 J^i(t, \mathfrak{X}) \tag{6}$$

Note that the equations split up into equations for the electric fields and for the magnetic fields separately, whic means that in the static limit, the theory splits into two separate areas, electric and magnetic.

eqnds 1' and 3 give

$$\frac{1}{\sqrt{g}}\partial_i \sqrt{g} g^{ij}\partial_j \Phi = \rho(\mathfrak{X}) \tag{7}$$

To begin let us first look at this equation in spherical polar coordinates.

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \tag{8}$$

with coordinates $x^1 = r; \quad x^2 = \theta; \quad x^3 = \phi$. Then

$$g_{11} \equiv g_{rr} = 1; \quad g_{22} \equiv g_{\theta\theta} = r^2; \quad g^{33} \equiv g_{\phi\phi} = r^2 sin(\theta)^2 \tag{9}$$

and all other components 0. Thus the metric matrix is diagonal and the determinant, g is $r^4 \sin(\theta)^2$

Thus the equation for Φ is

$$\frac{1}{r^2\sin(\theta)}\left(\partial_r r^2 sin(\theta) 1 \partial_r \Phi + \partial_\theta r^2 sin(\theta)\frac{1}{r^2}\partial_\theta \Phi + \partial_\phi r^2 \sin(\theta)\frac{1}{r^2\sin(\theta)^2}\partial_\phi\right)\Phi = -\frac{1}{\epsilon_0}\rho \tag{10}$$

Let us firstly assume that we have no θ or ϕ dependence in Φ, so the only derivative that survives is the r derivatives. This leads to

$$\frac{1}{r^2}\partial_r r^2 \partial_r \Phi(r) = -\frac{1}{\epsilon_0}\rho \tag{11}$$

Let us first assume that $\rho = 0$. Then multipy by r^2, integrate by r to get

$$r^2 \partial_r \Phi(r) = C \quad \text{a constant} \tag{12}$$

Dividing by r^2 and integrating, we get $\Phi(r) = -\frac{C}{r} + D$, another constant.

This solution has some weird features. The volume integral is

$$dV = \sqrt{g}dx^1 dx^2 dx^3 = r^2 \sin(\theta) dr d\theta d\phi \tag{13}$$

Using Gauss thm and recalling that the r component of E is perpendicular to the surfrace r=R const, we get

$$E_r = -\partial_r \Phi = \frac{C}{r^2} \tag{14}$$

$$\int g^{rr} E_r R^2 \sin(\theta) d\theta d\phi = \int \frac{\rho}{\epsilon_0} r^2 sin(\theta) dr d\theta d\phi \tag{15}$$

But $g^{rr}E_rR^2sin(\theta)d\theta d\phi = 4\pi R^2\frac{C}{R^2} = 4\pi C$ This tells us that the integral of rho must be non zero. But this equation is true no matter what the value of R is, including arbitrarily small. Thus the charge must be located at r=0, and the integral of ρ/ϵ_0must be non-zero and have value $4\pi C$.

The charge density must have the property that it is non-zero at only a single point, and and its integral over the volume must be $4\pi C$. But a fundamental property of integrals is that the integral over any function cannot depend on its value at a single point. This "thing" is called a distribtion, and is something which is zero everywhere except at a point, but its integral is finite. Mathematicians call this a distribution. It is something that is not a function, but something whose integral is non-zero. Since we are working in 3 dimensions, this is called a three dimensional delta function $4\pi C\delta^3(r)$. Ie, the δ^3 function is something which when integrated over the volume containing point where the argument of δ^3 is zero gives the value 1. The δ^3 thus has dimensions of $1/distance^3$). There is clearly nothing physical that could have these properties.

A one dimensional delta function is a distribution such that it 0 everywhere except at $x = 0$ but the integral over it is unity. It is called $\delta(x)$. nd example is $\frac{d\Theta(x)}{x}$ where $\Theta(x)$ is the Heaviside step function such that $\Theta(x) = 1$ if x¿0 and is 0 if x¡0. If we have a regular function $f(x)$ then

$$\int -a^a f(x)\frac{d\Theta(x)}{dx}dx$$

is independent of a. But integrating by parts we have

$$\int_{-a}^{a} f(x)\frac{d\Theta(x)}{dx}dx = \int_{-a}^{a}\frac{d}{dx}(f(x)\Theta(x)) - \int_{-a}^{a}\frac{df}{dx}\Theta(x)dx$$
$$= f(a) - \int_0^a \frac{df(x)}{dx}dx = f(a) - (f(a) - f(0)) = f(0) \tag{16}$$

Ie, the integral over any regular function times the delta function simply gives the value of the function at $x = 0$.

If one wants the delta function at any other point, we simply write $\delta(x - x') = \frac{d\Theta(x-x')}{dx}$

Let us change coordinates in the potential equation above.

$$z = rcos(\theta); \quad x = r\sin(\theta)\cos(\phi); \quad y = rsin(\theta)sin(\phi) \tag{17}$$

Since $\Phi(r, \theta, \phi) = \Phi(r(x, y, z), \theta(x, y, z), \phi(x, y, z)$ as it is a scalar, and as the equation is a tensor equation. The metric becomes

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{18}$$

and $r = \sqrt(x^2 + y^2 + z^2)$. The eqn 1 and 3 become

$$\partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi = \frac{-\rho(x, y, z)}{\epsilon_0} \tag{19}$$

and $\Phi = -\frac{1}{4\pi r} = -\frac{1}{4\pi\sqrt{x^2+y^2+z^2}}$

$$\partial_x^2\Phi + \partial_y^2 + \partial_z^2\Phi = -\delta(x)\delta(y)\delta(z) \tag{20}$$

Or if one centers r at other points,

$$\Phi = -\frac{1}{4\pi r} = -\frac{1}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{21}$$
$$\partial_x^2\Phi + \partial_y^2 + \partial_z^2\Phi = \delta(x - x')\delta(y - y')\delta(z - z') \tag{22}$$

Maxwell's equations are linear, so if we add two solution, including their charge densities, we still get a solution with charge density being the sum of the charge densities. In this case we have a solution whose charge density is the all located at a single point. By adjusting the amplitude of that at any point and summing over all points, one can get a solution with arbitrary charge density. Note also that integrals are just summations.

Thus if we do

$$\tilde{\Phi}(x, y, z) = \int \frac{1}{\epsilon_0 4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}(-\rho(x', y', z'))dx'dy'dz' \tag{23}$$

then $\tilde\Phi(x,y,z)$ obeys

$$\partial_x^2\tilde\Phi(x,y,z)+\partial_y^2\tilde\Phi(x,y,z)+\partial_z^2\tilde\Phi(x,y,z)=\delta(x-x')\delta(y-y')\delta(z-z')\frac{-\rho(x',y',z')}{\epsilon_0}dx'dy'dz'=-\frac{\rho(x,y,z)}{\epsilon_0} \tag{24}$$

Ie, we can solve the equation by use of what is called a Green's funtion integrated over the RHS of the equations. Ie, a devivative equaiton is turned into an integral equation. All linear equations with source have this behaviour. One can find a function, such that summing or integrating it over the source terms gives one a solution to the equation with the specific source.

Ie, we have

$$G(x,y,z;x',y',z')=\frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{25}$$