Div, Grad, Curl

I. ANTISYMMETRIC DERIVATIVE

Given and arbitrary cotangent type vector field $A_i(\mathfrak{X})$, the antisymmetric derivative $\partial_i A_j - \partial_j A_i$ is a two indes cotensor type tensor. Define

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{1}$$

If F_{ij} is a tensor then it must transform as

$$\tilde{F}_{ij} = \partial_{\tilde{i}} x^k \partial_{\tilde{j}} x^l F_{kl} \tag{2}$$

The right side of the equations transforms as

$$\begin{aligned} &\partial_{\tilde{i}} \tilde{A}_j - \partial_{\tilde{j}} \tilde{A}_i \\ &\quad = \partial\tilde{i}(\partial_{\tilde{j}} x^k A_k - \partial\tilde{j}(\partial_{\tilde{i}} x^k A_k) \\ &\quad = \Big[\partial\tilde{i}(\partial_{\tilde{j}} x^k) A_k - \partial\tilde{j}(\partial_{\tilde{i}} x^k) A_k\Big] \\ &\quad (\partial_{\tilde{j}} x^k \partial\tilde{i} A_k - (\partial_{\tilde{i}} x^k \partial\tilde{j} A_k) \end{aligned} \tag{3}$$

The part in square brackets is

$$\begin{aligned} &\Big[\partial\tilde{i}(\partial_{\tilde{j}} x^k) A_k - \partial_{\tilde{j}}(\partial_{\tilde{i}} x^k) A_k\Big] \\ &\quad = \Big[\partial\tilde{i}\partial_{\tilde{j}} x^k A_k - \partial\tilde{j}\partial_{\tilde{i}} x^k A_k\Big] \end{aligned} \tag{4}$$

But

$$\partial\tilde{i}\partial_{\tilde{j}} x^k = \partial\tilde{j}\partial_{\tilde{i}} x^k$$

which sets that equal to 0. The rest is just

$$\partial_{\tilde{i}} x^k \partial_j x^l (\partial_k A_l - \partial_l A_k) \tag{5}$$

which is the same as left side. Ie, this antisymmetric derivative of $A_i$is a tensor equation.

This means that the curl

$$\begin{aligned} B^i &= \frac{1}{2}\epsilon^{ijk}(\partial_j A_k - \partial_k A_i) = \frac{1}{2}\epsilon^{ijk}\partial_j A_k - \frac{1}{2}\epsilon^{ikj}(\partial_k A_j) \\ &= \epsilon^{ijk}(\partial_j A_k) \end{aligned} \tag{6}$$

transforms like a Tangent vector.

II. DIVERGENCE

The divergence has a similar problem. The divergence would look something like $\partial_i C^i$ But when one does a coordinate transformation, this would become

$$\partial_{\tilde{i}} C^{\tilde{i}} = \partial_{\tilde{i}} x^k \partial_k (\partial_l \tilde{x}^i C^l) = [\partial_{\tilde{i}} x^k \partial_l \tilde{x}^i \partial_l C^l \tag{7}$$

It is that second derivative term that is the problem Is there something using say the metric or the determinant that gets rid of this term? the ansswer is yes.

Define the divergence by

$$divC = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}C^i \tag{8}$$

We know that $divcurlC$ is supposed to be 0, but
$\partial_i(\epsilon^{ijk}\partial_j A_k)$ has an extra $1\sqrt{g}$ in the definiton of ϵ^{ijk}

$$\epsilon^{ijk} = \frac{1}{\sqrt{g}}e^{ijk} \tag{9}$$

where the coeficients of e are all ±1 or 0. Thus

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\epsilon^{ijk}\partial_j A_k) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\frac{1}{\sqrt{g}}e^{ijk}\partial_j A_k)$$
$$= \frac{1}{\sqrt{g}}e^{ijk}\partial_i\partial_j A_k = 0 \tag{10}$$

just as one knows from the behaviour of (div curl A) in cartesian coordinates. This thus gives the definiton of the div.

$$divC = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}C^i \tag{11}$$

III. GAUSS'S LAW

Having the expression for div and curl in an arbitrary coordinate system, we can now prove Gauss's thm and Stokes thm. byt evaluating them in a coordinate system which is specially selected to have certain properties. Then since we know that the equation is a "physical" expression (tensor) we know it must be true in an arbitrary coordiante system.

Lots look at the integral of the divergence of a tangent vector over a volume. We will asume that that the volume a simply connected region, as is the surface. We also assume that the surface is "smooth".

The integral of the divergence will be

$$G = \int_V \frac{1}{\sqrt{g}}\partial_i\sqrt{g}C^i(\sqrt{g}dx^1dx^2dx^3) \tag{12}$$

The term $(\sqrt{g}dx^1dx^2dx^3)$ is the physical volume element (length times width time height).

$$G = \int_V \partial_i(\sqrt{g}C^i)dx^1dx^2dx^3 = \int_V (\partial_2(\sqrt{g}C^1)dx^1)dx^2dx^3 + \partial_1(\sqrt{g}C^2)dx^1)dx^1dx^3 + (\partial_3(\sqrt{g}C^3)dx^3)dx^1dx^2 \tag{13}$$

Now, pick a point inside the volume, and define the coordinates to have the following features. At that point, x^1 has the value 0, and has the value 1 everywhere on the surface. Near the point choose, $x^1 = r, x^2 = \theta,\ x^3 = \phi$ with the metric corresponding to the these polar coordinates. We extend the line (θ, ϕ constants) from 0 to the surface arbitrarily but such that lines for different θ, ϕ never cross, and such that just at the surface, the lines hit the surface perpendiculary. Note that these lines of constant $\theta\phi$ need not be straight lines. They can wiggle all they want, just satisfying the above requirements. The second and third integrals along the lines of constant r, θ and r, phi are such that at their endpoints ($\phi = 0, 2\pi$) and constant r, θ are the same points in space, so the difference of C^3 at thos endpoints is 0. Similarly the integral over ϕ in the second integral goes to 0 at $\theta = 0, \pi$ so the seond and third integrals are both zero.

At r=1, becaue the lines are perpendicular to the surface, the g_{1i} components will be zero unless i=1. Thus the determinant will be of the form $sqrtg = \sqrt{g_11}\sqrt{^2g}$ where ^2g is the determinant of the 2-D metric on the surface. Thus the first term is

$$G = \int \sqrt{g_{11}}C^1(1, \theta, \phi)\sqrt{^2g}d\theta d\phi \tag{14}$$

But the length squared of the x^1 component of C is $g_{11}C^1$ and $\sqrt{^2g}d\theta d\phi$ is the physical surface element of the $x^1 = r$ surface. Ie the volume integral of the divergence is the integral over the surface of the length of the perpendicular compoent of C^i at the surface. Since this is the value in this coordinate system it must also be the answer in all coordinate systems.

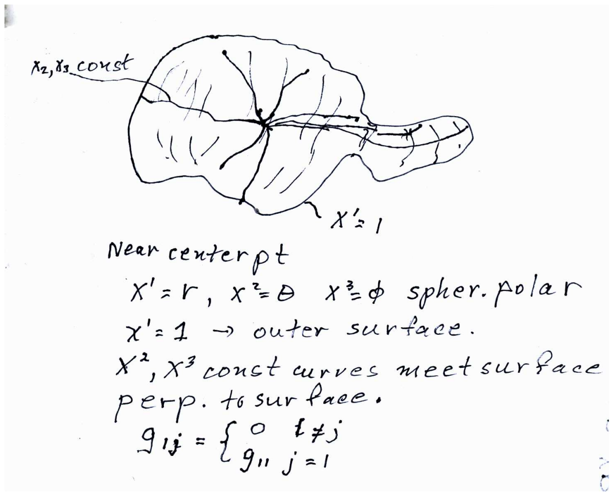

FIG. 1: Figure 1. Diagram for Gauss's law, with $x^1 = 0$ being some point inside the surface, $x^1 = 1$ being the outer boundary of region, $x1$ constant are topologically spheres (closed surfaces, with x^2, x^3 constant being the x^1 curves

IV. STOKES' THEOREM

Stokes them says that integral of the length of the perpendicular component of the curl over a surface equals the integral of the length of the component of the the vector parallel to the edge over the edge. Define the coordinate x^1 so that the surface to be defined by $x^1 = 0$ and assume as above that x^1 coordinate lines are perpendicular to the surface at the surface. Now layer the surface with a set of coordinates $x^2,\ x^3$ in a similar manner to the Gauss's law. Choose a point in the surface, and near that point define the coordinates x^2 and x^3 to be polar coordinates $x^2 = r,\quad x^3 = \phi$. Populate the surface with a set of non-crossing lines running from $x^2 = 0$ out to the edge of the surface. and define x^3 to be constant along these lines. and define x^2 so that it is 0 at the center point and is 1 at the boundary curve. Now define assume you have a cotangent vector field C_i such that its length everywhere is finite and such that the length of its curl is finite everywhere on the surface. Now integrate the length of the component of the curl perpendicular to the surface over the surface. This will be equal to the one dimensional integral along the boundary of the length of the component of the vector field itself along that boundary.

The curl of a cotangent type vector field C_i is $\epsilon^{ijk}\partial_j C_k$. The length of the component perpendicular to the surface is $\sqrt{g_1 1}\epsilon^{1jk}\partial_j C_k$. Because we constructed the coordinates so that the x^1 lines (ie the lines along which $x^2,\quad x^3$ are constant) are perpendicular to the surface at the surface, we have $g_{12} = g_{23} = 0$ and thus $g^{12} = g^{13} = 0$. The element of surface area of the surface is $\sqrt{{}^2g}dx^2dx^3$, the determinant of the metric restrited to the surface. At the surface, the integral of the length of the component of the curl perpendicular to the surface is thus be

$$\begin{aligned} S &= \int \sqrt{g_{11}}\frac{1}{\sqrt{g_{11}{}^2g}}e^{1jk}\partial_j C_k\sqrt{{}^2g}d\theta d\phi \\ &= \int e^{1jk}\partial_j C_k dx^2dx^3 \end{aligned} \tag{15}$$

Integrating with respect to x^2 in the first term and x^3 in the second, we get

$$S = \int_0^{2\pi} C^3dx^3|_{x^2=0}^1 - \int_0^1 C^2dx^2|_{x^3=0}^{2\pi} \tag{16}$$

The second term is 0 because the argument of has the same value at 0 and 2π for each x^2 (the line defined by $x^1 = 0,\quad x^2 - \text{const}$ is a closed curve).

The first term can now be written as

$$S = \int_0^{2\pi} \frac{1}{g_{33}}C_3\sqrt{g_{33}}dx^3|_{x^2=0}^1 \tag{17}$$

Since at $x^2 = 0$ the coordinates on the surface look like polar coordinates, g_{33}=0 and the length of the curve is 0. As long as the length of C_3 is finite, the contribuiton for $x^2 = 0$ is zero. Thus the only contribution is from the $x^2 = 1$ curve, which is the boundary of the surface by construction.

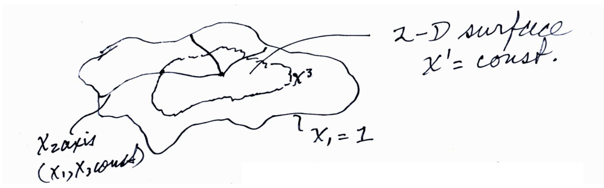

FIG. 2: Figure 2.

Diagram of coordinates for Stokes thm, with the $x^1 = 0$ defining the surface, and x^1 lines meeting surface perpendicularly, $x^2 = 0$ is point inside surface, $x^2 = 1$ is outer boundary, and the x^2 curves meeting the surface perpendicularly, and x^3 curves being closed curves with values from 0 to 2π.