

Electrostatics in media

For electrostatics in media, one does not use the full Maxwell equations, since the structure of the medium will in general have a high level of fluctuation of ρ and J with space inside the medium, and thus a high degree of fluctuation of the EM fields as well. Instead of taking into account all of these fluctuations, one looks instead at smoothed averages of these fields. Because of the linearity of electromagnetism, one can take averages, and the average of the density and currents will produce averaged fields $(\Phi, \vec{A}, \vec{E}, \vec{B})$. Given some averaging function $f(\vec{x} - \vec{x}')$, whose regime in this it is non-zero (or non-negligibly different from 0) is of the order of L which is much greater than the size of the atoms. Define the average $\rho, \langle \rho \rangle$

$$\langle \rho(x) \rangle = \int f(x - x') \rho(x') d^3 x' \quad (1)$$

Then we note that if we have any quantity, say $R(x)$, then

$$\partial_i \langle R(x) \rangle = \partial_i \int f(x - x') R(x') d^3 x' = - \int (\partial'_i f(x - x')) R(x') d^3 x' = - \int \partial'_i (f(x - x') R(x')) d^3 x' + \int f(x - x') \partial'_i R(x') d^3 x'$$

using integration by parts in the third expression, and that for large x' , $f(x - x')$ is zero. Thus the derivative of the local average of a function is the average of the derivative. This means that the derivatives of the average of the potential Φ is the average of the derivatives, and

$$\langle \nabla^2 \Phi(x) \rangle = \langle \nabla^2 \Phi \rangle(x) = -\frac{1}{\epsilon_0} \langle \rho(x) \rangle \quad (3)$$

$$\langle \Phi(x) \rangle = \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\vec{x} - \vec{x}'|} \langle \rho \rangle(x') d^3 x' \quad (4)$$

Lets assume that the material is made up of atoms, located at locations x_α^i , whose charge density for that specific atom dies on distances from the center of the atom of much less than L . Then we can write

$$\rho(x) = \sum_\alpha \rho_\alpha(x) \quad (5)$$

where ρ_α is the charge distribution of the atom α around the location x_α^i . Then we can write

$$\langle \rho(x) \rangle = \sum_\alpha \int f(x - x') \rho_\alpha(x') d^3 x' \quad (6)$$

$$= \sum_\alpha \int f(x - x_\alpha) + (x'^j - x_\alpha^j) \partial_{x_\alpha^j} f(x - x_\alpha) \rho_\alpha(x') d^3 x' \quad (7)$$

$$= \sum_\alpha (q_\alpha f(x - x_\alpha) - \partial_j P_\alpha^j) = \langle q(x) \rangle - \langle \partial_j P^j \rangle \quad (8)$$

[Note, the situation is actually more subtle than the above. In particular all those "atoms" or "molecules" are not independent entities, but they interact with each other. Those interactions create correlations between the atoms. (if one has a dipole moment, that dipole moment will interact with the charges in the next atom, creating correlations. Furthermore, except at absolute zero temperature, the atoms are moving and changing those correlations. Thus, the charge density distributions are not static but rather are dynamic and are also stochastic (the charge distributions, influenced by so many other atoms, cannot be said to have a specific distribution but a probability distribution over many different charge distributions. Thus one must also take statistical averages as well as spatial averages. Thus, that the phenomenological theory above works as well as it does in many situations is surprising, although that the averages are being taken over numbers like 10^{24} gives one at least the hope that it is a reasonable approximation.]

Thus we can approximate the mean charge density as the mean excess charge density of the atoms plus the mean divergence of the dipole moment of the atom. Thus we get the equation

$$\nabla \cdot \langle \vec{E} \rangle = \frac{1}{\epsilon_0} (\langle \rho(x) \rangle - \nabla \cdot \langle \vec{P} \rangle) \quad (9)$$

$$\nabla \cdot \langle \vec{E} \rangle + \frac{1}{\epsilon_0} \langle \nabla \cdot \vec{P} \rangle = \frac{1}{\epsilon_0} \langle q(x) \rangle \quad (10)$$

It is then usual to define

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (11)$$

and the equation becomes

$$\nabla \cdot \langle \vec{D} \rangle = \langle q(x) \rangle \approx \langle \rho_f \rangle \quad (12)$$

The $\langle q \rangle$ are called the "free" charges, while the $\nabla \cdot \vec{P}$ are called the bound charges. The nomenclature makes no sense, since the "free charges" are not necessarily free to move around, nor are the bound charges necessarily bound– they are just localized. Furthermore, the location of the charges, due to the interaction with neighbouring atoms, are correlated. Ie, if you have a fluctuation of the location of one atom, the fluctuations of neighboring atoms will be correlated (or in quantum terms, entangled) Griffith's worry about sodium chloride for example assumes that there are no fluctuation of the positions of the charges, which, unless the system were at absolute zero temperature classically, is not how the system behaves.)

[The "free charges" are assumed to be 0 for most situations. If not, the whole procedure becomes ill defined, since if a region has a net charge, then by altering the coordinate location that one takes as the location of the atom, one can make \vec{P} have any value one wants.

Consider a fixed charge distribution $\rho(x)$ supposedly centered at x_α . Then

$$Q = \int \rho(x) d^3x; \quad \vec{P}_\alpha = \int (\vec{x} - \vec{x}_\alpha) \rho(x) d^3x \quad (13)$$

Now instead of choosing x_α as the center, choose \vec{x}_β Then

$$Q = \int \rho(x) d^3x; \quad \vec{P}_\beta = \int (\vec{x} - \vec{x}_\beta) \rho(x) d^3x = \int (\vec{x} - \vec{x}_\alpha) \rho(x) d^3x + (\vec{x}_\alpha - \vec{x}_\beta) \int \rho(x) d^3x \quad (14)$$

$$= Q + \vec{P}_\alpha + (\vec{x}_\alpha - \vec{x}_\beta) Q \quad (15)$$

This means that we can make \vec{P}_β any value we wish, simply by choosing x_β appropriately. Mind you this will also make the higher multipoles change, and can make them become large enough that they contribute to $\langle \rho \rangle(x)$ substantially.]

Thus the electrostatic equations in a medium are

$$\langle E_i \rangle = -\partial_i \langle \Phi \rangle \quad (16)$$

$$\nabla \times \langle \vec{E} \rangle = 0 \quad (17)$$

$$\nabla \cdot \langle \vec{D} \rangle = \langle \rho_{\text{free}} \rangle \quad (18)$$

$$D = \epsilon_0 \vec{E} + \vec{P} \quad (19)$$

Note that this does NOT mean that \vec{D} is the gradient of a potential, since there is no particular reason except accident that $\nabla \times P = 0$ This is especially true if for example the material has edges. Since P can be non-zero only inside materials, if the material has an edge then the curl of P will probably not be zero.

Note also that if the material has an edge, then $\nabla \cdot \vec{P}$ will probably have a smeared (because of the averaging volume designated by f) delta function where P ends at the end of the material.

For most materials, for weak enough electric fields, the dipole moment will be linearly proportional to the electric field. For many atoms if one puts on an electric field, the atom will become polarized, as the electrons will be repelled by the field to the other side of the atom. This will in general be very small, because the $e\vec{E}d$ where d is the diameter of the atom, will be a far smaller energy than the binding energy of the electron to the atom. Only when the electric field becomes strong enough to strip electrons away from the atom, will the dipole moment become a non-linear function of the E field.

Depending on the internal structure of the atoms, the linear dependence can be positive (or rarely, and then almost always only at non-zero frequencies, negative).

$$\vec{P} = \chi \vec{E} \quad (20)$$

For positive χ it is called a dielectric material, while for negative χ it is a paraelectric material. Unlike for magnetism, ferroelectric materials are rare, but some materials when they are cooled below the melting point in a strong external electric field, can freeze-in any polarisation that was present in the melted material. These are called electrets, and just like permanent magnets, can be handy. For example the ultra cheap (but accurate) microphones have internal

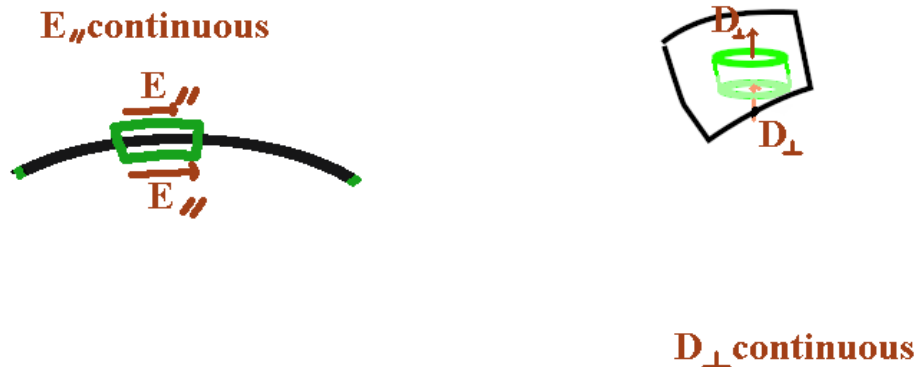


FIG. 1: Figure DE-cont.ps. Continuity via Stokes and Gauss

electrets to supply an electric field for a thin layer, usually of aluminized plastic held a little way from the top of the electret supplying an electric field at the "diaphragm". As the diaphragm is pushed toward the electret, the voltage between the the electret and the daphragm decreases ($V=EL$, where L is the distance between the electret and the diaphragm, and the change in voltage drives a change in current in the external circuit.

There is another feature of the above. We have

$$\nabla \times \vec{E} = 0 \quad (21)$$

as one of our equations. Now, starting at any point \vec{x}_0 , can draw a line from there to any other point \vec{x} . Now define the function

$$\Psi(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} \vec{E} \cdot d\vec{l}|_{\text{line1}} \quad (22)$$

, the line integral along the path from \vec{x}_0 to \vec{x} . If we take any other line between the two points, call it line 2, then the difference between the two integrals is the line integral of \vec{E} around the surface spanned by these two lines. But Stokes theorem says that this is the surface integral of $\nabla \times \vec{E}$ over that surface, which is 0, by the averaged Maxwell equations when $\langle \partial_t \vec{B} = 0 \rangle$. Thus the integral between the two points of \vec{E} is the same, no matter what the line taken is. Ψ is a function only of \vec{x} . Furthermore, by choosing the line so that near x the curve only goes in the x^i direction, we find that the derivative of Ψ in the x^i direction is uniquely defined as E_i . Thus,

$$E_i = \partial_i \Psi_i \quad (23)$$

and acts as a potential for E . (this was the same argument I used in the Aharonov Bohm note to argue that one could always find a gauge transformation which would set $\vec{A} = 0$ in a region where $\vec{B} = \nabla \times \vec{A} = 0$. This is not terribly helpful, since this says if we know what \vec{E} is, we can find a potential, but does not say what that potential is, or how to find it.

In the case where \vec{D} is linearly dependent on E , $\vec{D} = \epsilon \vec{E}$, and epsilon is a constant, then the $\rho_f = 0$

$$\nabla \cdot \vec{D} = 0 \quad (24)$$

and thus

$$\nabla \cdot \vec{E} = 0 \quad (25)$$

. Thus Ψ obeys

$$\nabla^2 \Psi = 0 \quad (26)$$

In the boundaries where ϵ changes we have the boundary conditions that \vec{D}_\perp is continuous. If we choose the surface to be $x^1 = \text{const}$, and such that $g^{11} = 1$ and $g^{1j} = 0$ for $j \neq 1$ then $\epsilon(x^1)\partial_1\Psi$ is continuous, as well as $\partial_j\Phi(\vec{x})$; $j \neq 1$ is continuous.

We will find that the same argument can be applied to H and B in the magnetic case. Ie, there to there are limited cases where H is also determined by a potential.

I. SHIELDING

Lets apply the above to dielectric shielding. Consider a sphere of outer radius r_2 and inner radius of r_1 , made of a dielectric material with very high dielectric constant $\epsilon/\epsilon_0 \gg 1$. Ie, between those two radii, the relation between \vec{D} and \vec{E} is

$$\vec{D} = \epsilon\vec{E} \quad (27)$$

Note that I am dropping the averaging symbols, knowing that if I am talking about D there must be averaging involved. Outside and inside we have the vacuum relation $\vec{D} = \epsilon_0\vec{E}$. Assuming that there are no "free" chages, The equations are

$$\nabla \times \vec{E} = 0 \quad (28)$$

$$\nabla \cdot D = 0 \quad (29)$$

In all three areas, since ϵ is constant, its derivatives are 0, so we have

$$\nabla \times \vec{E} = 0 \quad (30)$$

$$0 = \nabla \cdot (\epsilon\vec{E}) = (\nabla\epsilon)\vec{E} + \epsilon\nabla \cdot \vec{E} = \epsilon\nabla \cdot \vec{E} = 0 \quad (31)$$

so the potential within the region obeys

$$\nabla^2\vec{\Phi} = 0 \quad (32)$$

Since we have spherical symmetry, in each region we can expand the potential in spherical harmonics and get

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(C_{lm}r^l + D_{lm}r^{-(l+1)}) \quad (33)$$

where this came from our separation of variables solution of the spherically symmetric chargeless solution of the Poisson equaiton.

Now at the boundries, using Stokes thm on \vec{E} for a path which which hugs the surface of discontinuity of ϵ to get

$$\int_{path} \vec{E} \cdot d\vec{l} = \int_{surface} (\nabla \times \vec{E}) \cdot \vec{n} d^2\mathfrak{S} = 0 \quad (34)$$

so the interal over the path on the two sides of the surface are equal. Ie, the component of E parallel to the surface of discontinuity of dielectric permattivity is continuous.

But parallel to the surface of discontinuity of ϵ is perpendicular to the radius, and is thus is the θ and ϕ derivatives of Φ .

Thus $\partial_\theta\Phi$ and $\partial_\phi\Phi$ must be continous across the surface which means that $C_{lm}r_d^l + D_{lm}r_d^{-(l+1)}$ must be the same on either side of the surface even though the C an D coefficients will change on going across the surface. (r_d is the radius at which the dielectric constant is discontinuous).

Using Gausses thm on the radial derivative of Φ , since $\nabla \cdot \vec{D} = 0$ perpendicular to the surface of r equals constant, is $\epsilon\partial_r\Phi$ so we find that the radial component of D must be continuous. Thus

$$\epsilon\partial_r(C_{lm}r^l + D_{lm}r^{-(l+1)})\epsilon'\partial_r(C'_{lm}r^l + D'_{lm}r^{-(l+1)}) \quad (35)$$

at the value of r where ϵ changes. This must be true for each value of l, m .

Now place that sphere into a constant electric field, and lets assume that field points in the z direction. Then E_x, E_y are zero, and $E_z = E_0$, and Φ far way must be $-E_0z$. But $z = r \cos(\theta)$, and we have $\cos(\theta) \propto Y_{10}(\theta, \phi)$, so we

expect that only the Y_{10} spherical harmonic will enter the solutions. I.e, continuity of E_{parallel} implies that in all of the regions, the angular dependence is $\cos(\theta)$ Thus

$$\Phi = -(E_0 r + \frac{D_{10}}{r^2}) \cos(\theta); \quad r > r_2 \quad (36)$$

$$= (C'_{10} r + \frac{D'_{10}}{r^2}) \cos(\theta); \quad r_2 > r > r_1 \quad (37)$$

$$= (\tilde{C}_{10} r + \frac{\tilde{D}_{10}}{r^2}) \cos(\theta); \quad r < r_1 \quad (38)$$

But for $r < r_1$ as we go to zero, we want the potential to be regular, not singular, so $\tilde{D}_{10} = 0$. We thus have the 4 equations, two being that the parallel components of \vec{E} are continuous, and two that the perpendicular component of $D_r = \epsilon(r)E_r = -\epsilon(r)\partial_r\Phi$ is continuous.

$$(-E_0 r_2 + \frac{D_{10}}{r_2^2}) = C'_{10} r_2 + \frac{D'_{10}}{r_2^2} \quad (39)$$

$$C'_{10} r_1 + \frac{D'_{10}}{r_1^2} = \tilde{C}_{10} r_1 \quad (40)$$

$$\epsilon_0(-E_0 - 2\frac{D_{10}}{r_2^3}) = \epsilon(C'_{10} - 2\frac{D'_{10}}{r_2^2}) \quad (41)$$

$$\epsilon(C'_{10} - 2\frac{D'_{10}}{r_1^2}) = \epsilon_0 \tilde{C}_{10} \quad (42)$$

4 linear inhomogeneous equations in 4 unknowns can be solved. Of interest is the field inside. Solving for \tilde{C}_{10} which gives the potential inside the sphere, we get, using $\lambda = \frac{\epsilon}{\epsilon_0}$

$$\tilde{C}_{10} = -E_0 \left(\frac{9\lambda r_2^3}{2(r_2 - r_1)(r_1^2 + r_2 r_1 + r_2^2)(\lambda - 1)^2 + 9\lambda r_2^2} \right) \quad (43)$$

As λ gets very large, it is the first term in the denominator that dominates, And \tilde{C} becomes very small. The dielectric acts as shield for the inside of the cavity. (note that one could regard a conductor as a dielectric with $\lambda \rightarrow \infty$.)

(Note that pure water has a λ of about 80) so it acts as a good shield against static (and thus also low frequency) electric fields.)