Energy and forces

One of the questions about electromagnetism is the idea of energy and forces that the electromagnetic field exerts on matter. In many tradition approaches, this is attacked from a historical point of view– ie, what were the floundering insights that people had historically to arrive at the concept of energy of the electromagnetic field. On the other hand that is over. We have the concept.

$$\mathcal{E} = \frac{1}{2}\int_{\mathfrak{V}} \left(\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{\mu^0}\vec{B}\cdot\vec{B}\right) d\mathfrak{V} \tag{1}$$

$$= -\frac{1}{2}\int_{\mathfrak{V}} \left(\epsilon_0 g^{ij}E_iE_j + \frac{1}{\mu_0}g_{ij}B^iB^j\right) d\mathfrak{V} \tag{2}$$

is the energy in the electromagnetic field.

Let us stay in a coordinate system where the metric is the identity and $g = 1$, and in electro-magneto-statics. Then

$$\vec{E} = -\nabla\cdot\Phi \tag{3}$$

$$E_i = -\partial_i\Phi \tag{4}$$

$$\vec{B} = \nabla\times\vec{A} \tag{5}$$

$$B^i = e^{ijk}\partial_jA_k = e^{ijk}\partial_jA_k \tag{6}$$

Then we can write the equation for the energy as

$$\mathcal{E} = \frac{1}{2}\int_{\mathfrak{V}} \epsilon_0(-\partial^j\Phi)E_j + e^{ijk}\partial_jA_kB_i)d^3x \tag{7}$$

Doing an integration by parts we get

$$\mathcal{E} = \int_{\mathfrak{V}}\left[\partial_j\left(-\epsilon_0\Phi E^j + \frac{1}{\mu_0}e^{ijk}A_kB_i\right) - \left(-\epsilon_0\Phi\partial_jE^j + \frac{1}{\mu_0}e^{ijk}A_k\partial_jB_i\right)\right]d^3x \tag{8}$$

where all of the calculations have been done in Cartesian coordinates. ($g_{ij} = \delta_{ij}$ and $g = 1$)

The first term is the divergence of a vector field, and by Gauss's thm this equals the surface integral over boundary of the volume, which we take to be infinity, and assume that the fields fall off sufficiently fast that the integral over infinity is zero. Thus we are left with

$$\mathcal{E} = \int_{\mathfrak{V}}(\epsilon_0\Phi\partial_jE^j - \frac{1}{\mu_0}e^{ijk}A_k\partial_jB_i)d\mathfrak{V}$$

$$= \int_{\mathfrak{V}}(\rho\Phi + J^kA_k)d\mathfrak{V} \tag{9}$$

There is one thing which is a bit uncertain here, since A_k has a gauge freedom, such that $A_k = \tilde{A}_k + \partial_k\psi$ should not change the energy, since the enrgy is a physical quality, and should not change under a gauge transformation. But

$$\int_{\mathfrak{V}}(J^kA_k)d\mathfrak{V} = \int_{\mathfrak{V}}(J^k(\tilde{A}_k + \partial_k\psi)d\mathfrak{V} = \int_{\mathfrak{V}}(J^k(\tilde{A}_k) + \partial_k(\psi J^k) - \psi\partial_kJ^k)d\mathfrak{V} \tag{10}$$

But $\partial_kJ^k = 0$ and

$$\int_{\mathfrak{V}}\partial_k(\psi J^k)d\mathfrak{V} = \int_{\mathfrak{S}}\psi J^kn_kd\mathfrak{S} = 0$$

by Gauss's thm.i and assuming that J^i and ψ fall off suffciently fast at infinity. Thus $\int A_k J^k d\mathfrak{V} = \int \tilde{A}_k J^k d\mathfrak{V}$ as required to make the expression the same even when a gauge transformatin was carried out.

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \int_{\mathfrak{V}} \rho_i \Phi_2 + {J_1}^i A_{2i} d\mathfrak{V} \tag{11}$$

Ie, the interaction energy in the electromagnetic field in this static situation is just equal to the charge density of the one times the energy density of the other plus the current times the vector potential of the other. I looks as though one could regard the EM field as if it were just the non-local interaction of the charges of one system with those of the other. This is however simply a result of the static approximation were are working with.

I. FULL ENERGY

The above expression might give you the impression that there is no difference between the electric and mangnetic fields, and their sources. Ie, the Fields E and B are simply another way of writing the charges and the potentials as the source of the sources. While this confusion was not surprizing 150 years ago, that it still survives now is pretty astonishing. To see this, let us relax our approximation that the fields are time independent, and look at the full Maxwell equations Let us start with the energy of the EM field. Again we will work in Cartesian coordinates, where g_{ij} and g^{ij} are both the identity matrix, g is unity, and upper and lower index components are identical (eg, $E^i = E_i$ etc)

$$\mathcal{E} = \frac{1}{2}(\epsilon_0 E_i E^i + \frac{1}{\mu_0} B^i B_i) \equiv \frac{1}{2}(\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}) \tag{12}$$

Now take

$$\partial_t \mathcal{E} = \epsilon_0 (\partial_t E_i) E^i + \frac{1}{\mu_0}(\partial_t B_i) B^i \tag{13}$$

Using Maxwell's equations

$$\epsilon_0 \mu_0 \partial_t E^i = e^{ijk}\partial_j B_k - \mu_0 J^i \tag{14}$$

$$\partial_t B^i = -e^{ilm}\partial_l E_m \tag{15}$$

we get

$$\partial_t \mathcal{E} = \int \left(\frac{1}{\mu_0} E_i e^{ijk} \partial_j B_k - J^i E_i + \frac{1}{\mu_0} B_i e^{ilm} \partial_l E_m \right) d\mathfrak{V}$$

$$= -\int \frac{1}{\mu_0} e^{ijk} E_k (\partial_i B_k) E_j d\mathfrak{V} - \int J^i E_i d\mathfrak{V} - \frac{1}{\mu_0} e^{ijk} (\partial_i E_j) B_k dmf\mathfrak{V} \tag{16}$$

$$- \int_{\mathfrak{V}} \frac{1}{\mu_0} e^{ijk} \partial_j (E_j B_k) d\mathfrak{V} \tag{17}$$

$$\partial_t \mathcal{E} + \int J^i E_i d\mathfrak{V} = -\int_{\mathfrak{S}} e^{ijk} E_j B_k n_i d\mathfrak{S} \tag{18}$$

The term on the left is the flux of of $\frac{1}{\mu_0}\vec{E}\times\vec{B}$ into the surface. This would be the energy flux into the surface. The first term on the left is the increase in the EM energy. The second term is the energy pumped into the matter electric current by the EM field. This equation is the equation for the conservation of energy.

Thus the $\frac{1}{mu_0}\vec{E}\times\vec{B}$ is the energy flux, and this equation says that the flux of energy into the region equals the increase in energy in the region plus the work done by the EM field on the matter carrying the charge.

We can also define a momentum density $\mathcal{P}$. The energy flux has units of energy density times the velocity. The Momentum density will equal the mass density ($\rho_M = \mathcal{E}/c^2$) times the velocity, and so will be

$$\mathcal{P}_i = \frac{1}{\mu_0}\vec{E}\times\vec{B}/c^2 = \epsilon_0 \vec{E}\times\vec{B} \tag{19}$$

Then

$$\partial_t \mathcal{P}^i = \epsilon_0 e^{ijk}(\partial_t E_j)B_k + E_j \partial_t B_k \tag{20}$$
$$= e^{ijk}(\frac{1}{\mu_0} e_{jlm}(\partial^l B^m)B_k - J_j B_k) + \epsilon_0 E_j (e^{klm}\partial_l E_m) \tag{21}$$

Note that $e_{ijk}e^{ilm} = \delta^l_j \delta^m_k - \delta^m_j \delta^l_k$, so this becomes, recalling that $\nabla \cdot \vec{B} = \partial_j B^j = 0$ and $\nabla \cdot \vec{E} = \partial_j E^j = \frac{\rho}{\epsilon_0}$

$$\partial_t \mathcal{P}^i = -\rho E^i - e^{ijk} J_j B_k - \partial_j(\frac{1}{\mu_0}(B^i B^j) + \epsilon_0(E^i E^j)) + \frac{1}{2}\partial^i(\frac{1}{\mu_0} B_j B^j - \epsilon_0 E_j E^j) \tag{22}$$

The tensor

$$T^{ij} = \epsilon_0(E^i E^j - \frac{1}{2}E^k E_k \delta^{ij}) + \frac{1}{\mu_0}(B^i B^j - \frac{1}{2}B^k B_k \delta^{ij}) \tag{23}$$

is called the stress tensor of the electromagnetic field and represents the force in the i^{th} direction across the j^{th} surface. We finally get

$$\int_{\mathfrak{V}} (\partial_t \mathcal{P}^i) + \rho E^i + e^{ijk} J_j B_k d\mathfrak{V} = -\int T^{ij} n_j d\mathfrak{S} \tag{24}$$

Ie, the rate of change of the electromagnetic momentum, plus the change of the momentum of the matter equal the electromagnetic force acting across the surface of the volume we are concerned with.

This tells us the electromagnetic field exerts a force on the matter with charge ρ and current J^i equal to $\rho\vec{E} + \vec{J} \times \vec{B}$.

It also shows that the electromagnetic field exerts forces on adjacent EM fields.

Compare the force of the electromagnetic field on the matter with the usual Lorentz force law

$$m\vec{a} = q\vec{E} \ + \ q\vec{v} \times \vec{B} \tag{25}$$

or

$$\partial_t \vec{p} = q\vec{E} + q\vec{v} \times \vec{B} \tag{26}$$

where $\vec{p} = m\vec{v}$ is the momentum of the particle, where we have approximated the charge by a point charge distribution (a delta function) and $m\vec{a} = \partial_t \mathbf{p}$ the rate of change of the momentum of the particle. Ie, we define

$$\rho(\mathfrak{X}) = q\delta(x - X(t))\delta(y - Y(t))\delta(z - Z(t)) \tag{27}$$
$$J^x = \rho(\mathfrak{X})\frac{dX}{dt}; \quad J^y = \rho(\mathfrak{X})\frac{dY(t)}{dt}; \quad J^z = \rho(\mathfrak{X})\frac{dZ(t)}{dt} \tag{28}$$

Note that this limit is largely non-sense, because the EM energy of a delta function (E goes as $1/r^2$ and thus E^2 diverges as $\frac{1}{r^4}$ diverges, giving the point particle an infinite energy, which clearly makes very little sense.

The big conceptual problem is that the EM field will exert forces on the charges making up the matter, causing it to move around, and thus one needs equations telling one how the matter, and especially the charge distribution will move around (changing the charge and distributions) under the influence of the EM fields. Futhermore unless there are other non-EM forces on the matter, it will tend to fly apart (adjacent areas of the same charge will repel each other). Thus the answer will tend to depend on exactly what model one assumes for the matter. The equations of EM will also be highly non-linear (the EM field causes the matter to move differently which will cause the EM field near those charges to change,....)

Of course this does not really matter, since special relativity basically says that point particles make no sense anyway, and particles have to be replaced by fields, not particle physics. (This lesson is still a long way away from having been learned by most physicists).
