Potential with boundary values

Let us assume that we have system where on some closed surface we demand that the potential have a certain value on the boundary. Now consider the Green;s function $\tilde{G}(\mathfrak{X},\mathfrak{X}')$. Note that this is not the charge Green;s function, but the mathematical Green's function for the Poisson equation

$$\nabla^2\tilde{G}(\mathfrak{X},\mathfrak{X}') = \delta^3(\mathfrak{X};\mathfrak{X}') \tag{1}$$

Again we will be operating in CArtesian coordinates. Now cosider the expression

$$\int(\Phi(x)\nabla^2\tilde{G}(\mathfrak{X},\mathfrak{X}')) - \tilde{G}(\mathfrak{X},\mathfrak{X}')\nabla^2\Phi(x))d^3x = \Phi(x') - 0 \tag{2}$$

since we want to find the solution to $\nabla^2\Phi(x) = 0$ inside the region with bounary condition that $\Phi(\mathfrak{X})$ for the boundar of the region be a fixed set of potentials.

Then we find that

$$\begin{aligned}\Phi(x) \;&=\; \int_{\mathfrak{V}}\left(\vec{\nabla}\cdot(\Phi(x)\vec{\nabla}\tilde{G}(\mathfrak{X},\mathfrak{X}') - \tilde{G}(\mathfrak{X},\mathfrak{X}')\vec{\nabla}\Phi\right)d\mathfrak{V}\\ &\quad - \int_{\mathfrak{V}}\left(\vec{\nabla}\Phi\cdot\vec{\nabla}\tilde{G}(\mathfrak{X},\mathfrak{X}') - \vec{\nabla}\Phi\cdot\vec{\nabla}\tilde{G}(\mathfrak{X},\mathfrak{X}')\right)d\mathfrak{V}\end{aligned} \tag{3}$$

The last two terms cancel, while, Using Gauss's thm on the first terms we get

$$\Phi(x') = \int_{\mathfrak{S}}(\Phi(\mathfrak{X})\partial_n\tilde{G}(\mathfrak{X},\mathfrak{X}') - \tilde{G}(\mathfrak{X},\mathfrak{X}')\partial_n\Phi(x))d\mathfrak{S} \tag{4}$$

We now choose the function $\tilde{G}(\mathfrak{X},\mathfrak{X}') = G(\mathfrak{X},\mathfrak{X}') + F(\mathfrak{X},\mathfrak{X}') = 0$ for $\mathfrak{X}$ on the boundary $\mathfrak{S}$ of the region where we placed the boundary condition for Φ. (G is the usual Green's function (not the G_ρ function) and F obeys the sourceless Poisson equation and such that the sum makes $\tilde{G}$ zero everywhere on the boundary. Then the second term in the above expression for $\Phi(x')$ is zero on the bounday because $\tilde{G}$ is zero there. in the seond term $\Phi(x)$ is the boundary condition on Φ, which is given, and $\partial_n\tilde{G}$ can, at least in principle, be calculated for the expression for $\tilde{G}$. $\tilde{G}$ can be determined by the method of images that we used to find the potential for a charge located at the point $\mathfrak{X}'$.

This procedure allows us to determine the potential given the boundary conditions on Φ on the surface. Note that while the method of images was developed for volumes bounded by conductors, here we use exactly the same procedure even though the surface $\mathfrak{S}$ is NOT necessarily a conductor, but just some surface that we know what the boundary condition Φ is on that surface. Ie, while the method of images can have imaginary conducive surface, here the surface specified for the boundary conditions is treated as though it is an (imaginary) conductor.

This allows us to calculate the potential if we know what the value of the potential is on the surface.

Note that if we choose the green's function so that the normal derivative of $\tilde{G}$ is zero on the surface, and we have boundary condition such that the E field perpendicular to the surface is specified, the above also allow us to determine $\Phi(x')$ everywhere inside the region.

I. EXAMPLE

Let us take a surface $z = 0$ on which we set the potential to be V for $x < 0$ and -V for $x > 0$. We first need to find a Green;s function, $\tilde{G}(\mathfrak{X},\mathfrak{X}')$ which is zero at z=0 and which is and obeys $\nabla^2G(\mathfrak{X},\mathfrak{X}') = \delta(\mathfrak{X},\mathfrak{X}')$. We can use the method of images, to get

$$\tilde{G}(\mathfrak{X},\mathfrak{X}') = \frac{-1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right) \tag{5}$$

This is clearly zero when $z = 0$, and obeys the equation for $z > 0$.

The perpendicualr derivative to the surface is the derivative with respect to z, giving

$$\partial_\perp \tilde{G}(\mathfrak{X}, \mathfrak{X}')|_{z=0} = \partial_\perp \tilde{G}(\mathfrak{X}, \mathfrak{X}')|_{z=0} \partial_z \tilde{G}(\mathfrak{X}, \mathfrak{X}')|_{z=0} \tag{6}$$

$$= \frac{1}{2\pi} \frac{z'}{\sqrt{(x-x')^2 + (y-y')^2 + (z')^2}^3} \tag{7}$$

If we define $r^2 = (x-x')^2 + (y-y')^2$, then $dxdy = rdrd\phi$ (where $x - x' = rcos(\pi)$. This gives us

$$\int \partial_\perp \tilde{G}(\mathfrak{X}, \mathfrak{X}')|_{z=0} dxdy = \frac{1}{2\pi} \frac{z'}{\sqrt{z'^2 + r^2}} rdrd\phi = -\frac{z'}{\sqrt{z'^2 + r^2}}|_{r=0}^{\infty} = 1 \tag{8}$$

As $z' \to 0$, $\partial_\perp \tilde{G}(\mathfrak{X}, \mathfrak{X}')_{z=0} = 0$, but the integral goes to 1, so in that limit, $\partial_\perp \tilde{G}(\mathfrak{X}, \mathfrak{X}')_{z=0; z'=0} = \delta(x-x')\delta(y-y')$ Thus we that

$$\Phi(x', y', z' = 0) = \int \partial_\perp \tilde{G}(\mathfrak{X}, \mathfrak{X}')_{z=0; z'=0} \Phi(x', y', z' = 0) dx'dy' \tag{9}$$

Which is exactly what we want. (the value at $z' = 0$ of the potential is exactly equal to the predefined value of the potential as a function of x', y'.

Now for $z' > 0$, the value of the $\partial_\perp \tilde{G}$ at $x - x' = y - y' = 0$ is $\frac{1}{2\pi z'^2}$. For $r^2 = (x-x')^2 + (y-y')^2 >> z'^2$ the value goes as $\frac{z'}{r^3} << \frac{1}{r^2}$ so it dies rapidly to zero. Thus the width of the function is or order $r \approx z'$. Thus the larger z' is, the more $\Phi(x', y', z')$ is a smearing out of the intial potential over a sphere or order z'.

II. SPHERICAL BOUNDRY.

One can follow the same procedure for a spherical cavity. The Green's function is

$$G(\mathfrak{X}, \mathfrak{X}') = -\frac{1}{4\pi\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \tag{10}$$

We want to add to this a solution to the source free equation inside the cavity, and this can be done using the method of images. The strength of the source outside the radius R of the cavity has radial position of $\frac{r''=R^2}{r'}$ and strength $-R/r'$. If we assume that the center of the sphere is at $x = y = z = 0$ this gives the image charge Green's function

$$G_I(\mathfrak{X}.\mathfrak{X}'') = \frac{R}{r'} \frac{1}{4\pi \frac{\sqrt{((x - x'\frac{R^2}{r'^2})^2 + (y - y'R^2}}{r'^2})^2 + (z - z'\frac{R^2}{r'^2})^2} \tag{11}$$

so that

$$\tilde{G}(\mathfrak{X}, \mathfrak{X}') = G(\mathfrak{X}, \mathfrak{X}'') + G_I(\mathfrak{X}.\mathfrak{X}'') = 0 \tag{12}$$

if $\mathfrak{X}$ lies on the circle R. Ie, if we take

$$\tilde{x} = x\frac{R}{r}; \quad \tilde{y} = y\frac{R}{r}; \quad \tilde{z} = z\frac{R}{r}; \tag{13}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ Now we can simplify this by choosing $y' = z' = 0$. Ie we are chosing the poi

So

$$\tilde{G}(\mathfrak{X}, \mathfrak{X}') = \frac{1}{4\pi} \left(\frac{-1}{\sqrt{(x\frac{R}{r} - x')^2 + (y\frac{R}{y} - y')^2 + (z\frac{R}{r} - z')^2}} + \frac{R}{r'} \frac{1}{\sqrt{(x\frac{R}{r} - x'')^2 + (y\frac{R}{y})^2 + (z\frac{R}{r})^2}} \right) \tag{14}$$

$$= 0 \tag{15}$$

where $x'' = x'\frac{R^2}{r'^2} = \frac{R^2}{x'}$ Ie, this makes the potential $\tilde{G}(\mathfrak{X}_{r=R}, \mathfrak{X}')$ on the surface $r = R$ zero.

An interesting situation happens when $x' = y' = z' = r' = 0$.Then G_I goes to a constant, namely $\frac{1}{4\pi R}$. Thus

$$\partial_\perp G(\mathfrak{X}, 0)|r = R = \partial_r \frac{1}{4\pi R^2} \tag{16}$$

WE thus have

$$\Phi(0) = \int_{r=R} \partial_\perp G(\mathfrak{X}, 0)|r = R\Phi(\mathfrak{X}) R^2 sin(\theta) d\theta d\phi = \int_{r=R} \Phi(\mathfrak{X}) \frac{sin(\theta) d\theta d\phi}{4\pi} \tag{17}$$

We thus have that the value of the field Φ at some point is equal to the average value of Φ on a sphere of any radius surrounding that point. This is called the Mean Value Thm for a potential which satisfies Poisson's equation.

iAn easier way to do this for the value of the field at the center of the sphere is to note that if $\mathfrak{X}'$ is at the center of the sphere (which we can call $\mathfrak{X}' = 0$, is to note that the value of the $G(\mathfrak{X}, \mathfrak{X}')$ with $\mathfrak{X}' = 0$ is $G(\mathfrak{X}, 0) = \frac{-1}{4\pi r}|_{r=R} = -\frac{1}{4\pi R}$ which is a constant. A solution of Poincare's equation is that the potential is constant, so if we take $G_I(\mathfrak{X}, 0) = \frac{1}{4\pi R}$ we would have

$$\tilde{G}(\mathfrak{X}, 0) = \frac{-1}{4\pi r} + \frac{1}{4\pi R}. \tag{18}$$

Then

$$\partial_\perp \tilde{G}(\mathfrak{X}, 0)|_{r=R} = \frac{1}{4\pi r^2}|_{r=R} = \frac{1}{4\pi R^2} \tag{19}$$

and

$$\Phi(0) = \frac{\int \Phi(R, \theta, \phi) dS}{\text{Area of } \mathfrak{S}} \tag{20}$$

Ie, again the value of the potential at the midpoint of the sphere equals the average value of the potential over the surface of that sphere.
