Energy and forces

I. MAGNETISM

The equations for the magnetic field are

$$B^i = \epsilon^{ijk}\partial_j A_k$$
or for cartesian
$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{1}$$
$$\tag{2}$$

and

$$\epsilon^{ijk}\partial_j B_k = \mu_0 J^i \tag{3}$$
or
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \tag{4}$$

The vector potential has a "guage" transoformation

$$A_i \to A_i + \partial_i\psi$$
$$\vec{A} \to \vec{A} + \vec{\nabla}\psi \tag{5}$$

which leaves $\vec{B}$ unchanged. In particular one can find a transformation such that the transformed $\vec{A}$ is divergence free.

$$\nabla\cdot\vec{A} + \nabla\cdot\nabla\psi = 0 \tag{6}$$

We would need to find

$$\nabla\cdot\nabla\psi = \nabla^2\psi = -\nabla\cdot\vec{A}. \tag{7}$$

This has as a solution, using the Green's function

$$\psi(\mathfrak{X}) = -\int G(\mathfrak{X},\mathfrak{X}')\nabla\cdot\vec{A}(\mathfrak{X}')d\mathfrak{V}$$
$$= \int \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\partial_i A^i dx'dy'dz' \quad) \tag{8}$$

Note that $\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ is often written as $|x-x'|$ or $\vec{x}-\vec{x}'|$.

Since we can always find such a solution as long as $\vec{\nabla}\vec{A}$ falls of sufficiently fast as one approaches spatial infinity,, we are thus going to assume that $\vec{\nabla}\cdot\vec{A} = 0$.

writing the equation for B in terms of A we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0 J \tag{9}$$

or

$$\epsilon^{ijk}\partial_j(\epsilon_{klm}\partial^l A_m) = \mu_0 J^i \tag{10}$$

(where again we are working in Cartesian coordinates). Then, recalling that

$$\epsilon^{ijk}\epsilon_{klm} = \epsilon^{kij}\epsilon_{klm} = \delta^i_l\delta^j_m - \delta^j_l\delta^i_m \tag{11}$$

we get

$$\partial^i \partial^j A_j - \partial^j \partial_j A^i = \mu_0 J^i \tag{12}$$

or

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \tag{13}$$

But we have chosen the gauge so that the first term is 0, which finally gives

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{14}$$

For each component of $\vec{A}$– ie, A^i this is just the same as the equation for the scalar potential of the electric field as a function of the density (with $\frac{1}{\epsilon_0} \rightarrow \mu_0$) (also $\vec{\nabla}(\nabla^2 \vec{A}) = \nabla^2(\vec{\nabla} \cdot \vec{A}) = -\mu_0 \vec{\nabla} \cdot J = 0$ which means that $\vec{\nabla} \cdot \vec{A} = 0$ is the unique solution of this equation.

Note that this is only the solution if the A^i are the cartesian components. Any other coordinate components are much more complicated.

Thehe assumption we made abount $\vec{\nabla} \cdot \vec{A} = 0$ is consistant with the solution found for $\vec{A}$ because J is divergence free..

Let us now look at some proplerties of the integrals over J.

$$\int \partial_i x^j J^i d\mathfrak{V} = \int (\delta_i^j J_i + x^i \partial_i J^i) d\mathfrak{V} \tag{15}$$

The left hand side equals the integral over the surface at infinity, by Gauss' law, and is zero as long as $\vec{J}$ falls off rapidly enough at infinity. The first term in the on the RHS is just the integral over each compoent J^j of the current, and the second term is zero because the divergence of the current is 0 in magnetostatics. Thus we have

$$\int \vec{J} d\mathfrak{V} = 0 \tag{16}$$

We can do the same thing for

$$\int \partial_i (x^j x^k J^i) = \int \delta_i^j x^k J^i + \delta_j^k J^i + x^j x^k \partial_i J^i) d\mathfrak{V} \tag{17}$$

again if J falls off sufficiently fast (at least as $1/r^4$) at infinity, the left side is zero, and the last term on the right is 0 because $\nabla \cdot J = 0$.Thus

$$\int (x^k J^j + x^j J^k) d\mathfrak{V} = 0 \tag{18}$$

Ie,

$$\int x^j J^k d\mathfrak{V} = \frac{1}{2} \int (x^j J^k - x^k J^j) d\mathfrak{V} \tag{19}$$

which means that the integral of $\vec{x}$ times $\vec{J}$ is antisymmetric, like the curl.

Define the magnetic dipole as

$$\vec{M} = \frac{1}{2} \int \vec{x} \times \vec{J} d\mathfrak{V} \tag{20}$$

as the dipole moment of the magnetic field, just as one defined the $\vec{P} = \int \vec{x} \rho d\mathfrak{V}$ as the dipole moment of the charge source of the Electric field.

If we look at the Electric and Magnetic fields, we get

$$\Phi(\mathfrak{X}) = \int \frac{1}{4\pi\epsilon_0 \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \rho(\mathfrak{X}') d\mathfrak{V}' \tag{21}$$

$$A_i(\mathfrak{X}) = \int \frac{\mu_0}{4\pi \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} J_i(\mathfrak{X}') d\mathfrak{V}' \tag{22}$$

Let us assume that $\mathfrak{X}$ is much larger than any $\mathfrak{X}'$, and define $r = \sqrt{(x)^2 + (y)^2 + (z)^2}$

$$\frac{1}{4\pi\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \approx \frac{1}{4\pi}\left(\frac{1}{r\sqrt{1 - 2\frac{(xx'+yy'+zz')}{r^2} + O(1/r^2)...}}\right)$$
$$\approx \frac{1}{4\pi}\frac{1}{r}(1 + \frac{(xx'+yy'+zz')}{r^2} + O(\frac{1}{r^3})). \tag{23}$$

since x, y, z are of order r. Thus

$$\Phi \approx \frac{1}{4\pi\epsilon_0}\int \rho(x')(1 + \frac{x^i x'_i}{r^2})d\mathfrak{V}' \approx \frac{-1}{4\pi\epsilon_0}(\frac{Q}{r} + \frac{\vec{x}\cdot P}{r^3}) \tag{24}$$

and

$$A_i \approx \frac{\mu_0}{4\pi}\frac{(\vec{x}\times\vec{M})_i}{r^3} \tag{25}$$

where in each case we have neglected terms which fall off faster than the dipole terms.

A. Energy

We found that in statics, the interaction energy can be written as

$$\mathcal{E}_I = \int \vec{E}_1 \cdot \vec{E}_2 d\mathfrak{V} = \int \rho_1(\mathfrak{X})\Phi_2(\mathfrak{X}) + J_1^i(\mathfrak{X})A_{2i}d\mathfrak{V} \tag{26}$$

where we have assumed that the densities and current are concentrated around $x_0, y_0, z_0 \equiv \mathfrak{X}_0$. We assume that the scalar and vector potential does not change rapidly aroung $\mathfrak{X}_0$ so we can expand them a taylor series around that point. We then get

$$\mathcal{E}_I \approx \int \left(\rho_1(\mathfrak{X})(\Phi_2(\mathfrak{X}_0) + (x^i - x_0^i)\partial_i\Phi_2(\mathfrak{X}_0) + ...\right) d(\mathfrak{V})$$
$$+ \int \left(J_1^i(\mathfrak{X})A_{2i}(\mathfrak{X}_0) + J_1^i(\mathfrak{X})(x^j - x_0^j)\partial_j A_{2i}(\mathfrak{X}_0)\right) d\mathfrak{V}$$
$$= \left(Q(\Phi_2(\mathfrak{X}_0)) - P_1^i E_{2i}(\mathfrak{X}_0) + ...\right) + \left(M_1^i B_{2i}\right) \tag{27}$$

where I have used that the integral over J is zero as shown above, and that the integral over the coordinate-times-the-current is antisymmetric.

Let us assume that the total charge Q is zero (a neutral charge distribution).

We note that the signs of the two dipole terms, the electric and magnetic terms, are opposite. If $\vec{P}$ and $\vec{E}$ are alined, the energy in the electromagnetic field decreases while if $\vec{M}$ and $\vec{B}$ are aligned, then the interaction energy increases.

On the other hand, the force between two aligned electric and between two aligned magnetic dipoles is attractive.

$$F_S^i = \int_{\mathfrak{V}_S} \rho(\mathfrak{X})_S E_{iD}(\mathfrak{X}) + \vec{J}_S \times \vec{B}_D d\mathfrak{V}_S$$
$$= P_S^j \partial_j E_i + M_S^j \partial^j B_D^i \tag{28}$$

The result is similar under the interchange of S and D. Since E and B change sign under that interchange, and by assumption, P and M do not, the force on $\vec{F}_S = -\vec{F}_D$ as required by Newton's third law.

$\mathfrak{V}_D$ is a small volume surrounding the dynamic dipole, while $\mathfrak{V}_S$ surrounds the static dipole. Note that the form of the contribution from the electric and magnetic dipoles are similar. One might thus jump to the conclusion that the work done by F_D should be the same for the electric and Magnetic term. However, this is not the case because the electric and magnetic forces behave very differently. From the equations for the change in energy of the system,

$$\partial_t\mathcal{E} + \vec{J}\cdot\vec{E} = -\frac{1}{\mu_0}\vec{\nabla}\vec{E}\times\vec{B} \tag{29}$$

The term $\vec{J} \cdot \vec{E}$ is the flux of energy out of the electromagnetic field into the mechanical degrees of freedom of the dipole mechanical system. It is important that it is only the E field that interacts with the charges and the currents and thus only the E field that can transfer energy between the EM field and the other degrees of freedom of system carrying the charges and currents. This is not surprizing since the force on charged matter due to the B field goes as $\vec{J} \times \vec{B}$, which is perpendicular to the current, and thus cannot feed energy into the matter.

In our case, the energy fed into the EM field of the dipoles is opposite sign for the magnetic and electric cases.

In the case of the stationary S magnetic dipole, it sees the $\vec{B}_D$ field changing and we know from Maxwell's equaitons that $\partial_t \vec{B}_D = -\vec{\nabla} \times E_D$ so the S dipole sees an electric field which opposes the $\vec{J}_S$ current. In order to hold the current constant, and thus keep the dipole moment constant, it has to feed energy into the current. But this energy can only come from the internal mechanical energy of the static magnetic dipole carrier, as otherwise the current would have to decrease. This leads to the energy of static dipole increasing. The net result is that the work done due by the force on the dynamic dipole, instead of decreasing the energy of the coupled system, that energy instead increases due to the conversion of the internal energy of the static dipole into increasing the current.

We will be interested in the forces acting between a Static (S) dipole and a Dynamic(D) dipole, where the dynamic one moves very slowly (quasi-statically). I will assume that the dipoles are all pointed in the same direction, and that the line connecting their centers $\mathfrak{X}_0$ is also along the same direction. Lets assume that this is z axis.

One would expect that this should result in a decrease in the energy in both cases, since in both cases, the two parallel dipoles exert forces on each other drawing them together, and if that occurs slowly enough, this would mean that the forces extract energy from the system. It would seem that the only source of energy is the electromagnetic field, and thus the energy in the electromagnetic field should decrease and energy is extracted. While true of the electric dipoles case, it is not true in the magnetic case. In that case we will find that the magnetic dipole must have internal "mechanical" or "chemical" degrees of freedom, which also carry energy. While the total EM and mechanical energies must decrease because of the work done on the forces, the energy in the EM field alone increases. The energy extracted by the work done by the attractive forces, and energy deposited into the EM field must come from those internal degrees of freedom..

In the case of the electric dipole, if the E field increases, the only thing that happens is due to the interaction of the two charges with the E field. As the first dipole moves in the field of the second, no internal energy is needed to keep the two charges making up the dipole apart. While the force might increase, the distance between the two charges (modeling the electric dipole as two opposite charges held a fixed distance apart), the E field does not work in keeping the dipoles apart. THey do not move with respect to each other.

On the other hand in the magnetic case, as the B field increases as the two dipoles move together , that time rate of change of B produces an electric field via the Maxwell equation $\partial_t B = -\nabla \times E$. The electric field decreases J, unless the system itself pumps in more energy to keep J as large as it was.

Let us look at the process in detail in a specific simple model. The model, which I will call the wheel model (where I imagine the wheel is a bicycle wheel). On the surface of the rim, I fasten a uniform thin layer of charges, and set the wheel rotating around its axis. That will give the wheel an angular momentum (with energy $\frac{1}{2} I\omega^2$ with I the moment of inertia about the axle, and ω the angular velocity). The wheel had radius R. The distribution of charge and current of

$$\begin{aligned}
&\rho(x) = \rho_0(\delta(r-R)\delta(z-z0) - 2\pi delta(x)\delta(y)\delta(z-z0)) \\
&J_z = 0 \\
&J_x = -- J_0 \frac{y}{R}\delta(r-R)\delta(z-z0) \\
&J_z = J_0 \frac{x}{R}\delta(r-R)\delta(z-z0)
\end{aligned} \tag{30}$$

where $J_0 = \rho_0 \omega R$. The charge at the center of wheel is to make sure that wheel is neutral, and that the electric field of the two charge charge distristributions can be neglected. The charge distribution has a lowest moment of the a quadrapole moment.

The difference between the two wheels making up the dipole is that for the stationary wheel, $z0 = 0$ while for the wheel whose center of mass we move, it is $z_0 = Z(t)$. We will ultimately allow the dynamic wheel to move, ie to have $Z(t)$ be a function of time, with $dZ/dt = -|u|$ where u is very close to zero (adiabatic motion). In order to allow the dipole to come closer together, we will clearly need to allow it to move for a longer period, the smaller u is.

The vector potential due to the current J_S at $z = 0$ near the dynamic loop near $z = Z$

$$A_{Si} = \int \frac{\mu_0}{4\pi} \frac{1}{\sqrt((x-x')^2 + (y-y')2 + (z-z')^2}} J_{Si}(\mathfrak{X}') d^3\mathfrak{X}' \tag{31}$$

We assume that $z \approx Z >> (x, y, R)$ and are thus interested in the field far away from the loop. We will be intergrating Writing

$$(x-x')^2+(y-y')2+(z-z')^2 = ((x^2+y^2)+(x'^2+y'^2)-2(xx'+yy')+(z-z')^2 \quad (32)$$
$$= R^2+z^2+2(xx'+yy') \quad (33)$$

and using that $z'=0$ for the loop, we have

$$\frac{1}{\sqrt{R^2+z^2-2(xx'+yy')}} = \frac{1}{\sqrt{z^2+R^2}\sqrt{1-2\frac{xx'+yy'}{z^2+R^2}}}$$
$$\approx \frac{1}{|z|}+\frac{xx'+yy'}{|z^3|+O(|z|^{-5}} \quad (34)$$

since we are interested in $z \approx Z >> (x,y,R)$. The absolute values come from taking the positive root of $\sqrt{z^2+R^2}$. Since $|z|>>R$, we can also approximate $\sqrt(z^2+R^2)by|z|$

$$A_{Si} \approx \frac{\mu_0}{4\pi}\left[\frac{1}{|z|}+\frac{x^j x'_j}{|z|^3}+...\right]J_i(\mathfrak{X}')d^3x' \quad (35)$$

The first term in square brackets gives 0 for the integral while the second term is a non-zero. Since $x'=R\cos(\phi')$, $y'=R\sin(\phi')$, $dx'dy'=r'dr'd\phi'$, we have

$$A_{Sz} = 0$$
$$A_{Sx} = \frac{\mu_0}{4\pi}\frac{1}{|z|^3}[\int_0^{2\pi} xR^2cos(\phi')sin(\phi)(-J_0)+yR^2sin(\phi')^2\rho_0\omega)Rd\phi'$$
$$= -y\frac{\mu_0}{4\pi|z^3|}J_0(\pi R^2)$$
$$A_{Sy} = x\frac{\mu_0}{4\pi|z^3|}J_0(\pi R^2) \quad (36)$$

$\rho_0\omega R\pi R^2 = J_0$ Area is the magnitude of the magnetic dipole moment M of the current loop, and $\vec{A}_S$ at all points around the rim of the second wheel are parallel to $\vec{J}_D$ the current at the loop at $z=Z$.

From $\vec{A}_S$ we have $\vec{B}_S = \vec{\nabla}\times\vec{A}_S$. This leads to

$$B_{Sz} = \partial_x A_{Sy}-\partial_y A_{Sx} = 2\frac{J_0\mu_0}{4\pi|z|^3}\pi R^2$$
$$B_{Sx} = \partial_y A_{Sz}-\partial_z A_{Sy} = 3\frac{J_0\mu_0}{4\pi z|z|^3}x\pi R^2$$
$$B_{Sy} = \partial_z A_{Sx}-\partial_x A_{Sz} = 3\frac{J_0\mu_0}{4\pi z|z|^3}y\pi R^2 \quad (37)$$

Note that I have written $\partial_z|z^\alpha| = \alpha z|z|^{\alpha-2}$ for odd α. Note that the x,y components of B are radial and it is these which will produce the forces on the dynamic dipole. The z component of the magnetic field will produce a radial force which will not necessaril produce any effect on the mechanical energy of the wheel as that force is can be neutralised by the radial forces in the spokes of the wheel.

For the other dipole D its effect at $Z=0$one has a similar expression, but with $Z\to -Z$ and $z\to z$. The z component of the B field will not change, while the x and y components will switch sign. Now, the dipole moment of the second loop is the same as the first, since they are identical. Note that I have assumed that both wheels are identical in their parameters.

We will asume that the top one is moving toward the bottom one ($\frac{dZ}{dt}=-u$).If this is the case, then the there is an extra current, $J_{Dz}=\rho_0\frac{dZ}{dt}\delta(r-R)\delta(z-Z)$. The force due to the magnetic field of the bottom loop on the top loop will be

$$\vec{F} = \vec{J}_D\times\vec{B}_S \quad (38)$$

The component of the force due to B_z, which is radial and orthogonal to both z axis and to the $\vec{J}$ will be in the radial direction which has no effect on the energy balance. It is only the radial component of B_D will be important and it will exert a force on J_D in the z direction. Since the sationary one is not moving, this for will do no work, and change no energies.

Let us look at the dynamic dipole again. Calling J_t, the tngential or axial current, and J_z the z component of the current, we find that both produce forces identical in magnitude and identical in the work they do on the wheel. There is no energy that they transfer to or from the EM field of the other loop. The work done by the z force on the outside mechanical world (the rope) equals the energy transfered from the mechanical rotation of the wheel.

The ratio of the two currents, the axial current dirven by the rotating rim, and the z-current carried by the center of mass motion are $\omega R/\frac{dZ}{dt}$. The ratio of the velocities of the currents (which, since both currents are orthogonal to the radial B_S field) is $\frac{dZ}{dt}/omegaR$ Thus the each does an equal amount of work. Thus, the work done by the magnetic field in each case is the same. However in one case the work is extracted from the internal energy of the dipole, while in the other it is the work delived out to the mechanism slowly lowering the dipole. This makes since as it is the same chanrge in both cases whose motion creates the currents, and it is the same radial component of the B_D. The net force is perpendicular to the total current and thus cannot do any work on that total force, It can simply transfer work from one type to another.

One might have imagine that the work done by the force pulling the two together is can also be expressed as the work done as one brought the two systems together. would have come from the EM field. This is certainly true in the Electrical case, and in fact almost all textbooks justify the energy in the EM field by showing that the energy released or needed in bringing two systems together is equal to the work done by the forces needed to bring them together in the limit as one brings them together vary slowly for the electric dipole case. (Doing it quickly would in addition release radiation, which would not be captured in the above energy budget).

However, while true of the electrostatic case, it is not true of the magnetic case. The answer in that case is that the energy gained/lost by bring the systems together slowly is the negative of the energy stored in the EM field. The quasistatic argument fails spectacularly. in the magnetic case.

Instead the energy transfered to the external non-electromagneitc world comes from the internal mechanical energy of the dipole.

However in the above we have just acconted for the energy extracted from the system by slowly lowering the upper wheel. We have not accounted for the increase in energy of the Electromagnetic field associated with the wheels.

This energy comes from the lower wheel. Since it is held fixed, it cannot come from the motion of the center of the wheel. There is no z component of the motion, so the energy cannot come from work that the center of mass does on the external world. Nor can it come from the magnetic force due to the current created by the motion of the center of mass of the wheel.

The answer is the magnetic field of the upper wheel at the stationary lower wheel is changing. due to the motion of the upper wheel. The upper magnetic field at the lower wheel is time dependent. Maxwell's equations tell us that

$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}. \tag{39}$$

The easiest way to calculate this electric field is to note that in addition to B changing in time, $\vec{A}$ does as well, and that in the absence of electric potentials, and because the onlt time dependence in our case is via Z

$$\vec{E} = -\partial_t \vec{A} = -\frac{dZ}{dt}\partial_Z \vec{A}. \tag{40}$$

Since A is axial, as shown above, so is $\vec{E}$ caused by the motion of the upper dipole. Also,since $\frac{dZ}{dt}$ is negative, $\partial_Z \vec{A}$ is proportional to $-\vec{A}$ since the vector potential depends on Z to an inverse power, the Electric field is proportional to $-\vec{A}$, and since $\vec{J_S}$ is proportional to $\vec{A}_D$, the work done by the electric field transfers energy from the mechanical innards of the static monople to the electromagnetic field.

$$\begin{aligned} E_z &= \frac{dZ}{dt}\partial_Z A_z = 0 \\ E_x &= \frac{dZ}{dt}\partial_Z A_x = 3\frac{dZ}{dt} y \frac{\mu_0}{4\pi|Z^4|} M \\ E_y &= \frac{dZ}{dt}\partial_Z A_x = -3\frac{dZ}{dt} xy \frac{\mu_0}{4\pi|Z^4|} M \end{aligned} \tag{41}$$

Thus the magnetic dipole not only transfer energy from the inner energy of the top monople to the "rope" lowering that dipole, and from the bottom inner energy to the surrounding EM field. For the electric dual dipoles, there is no available internal energy (since the struts holding apart the two charges of the dipole are assumed not to do any work) there is only the transfer of energy from the electromagnetic field to the "rope".

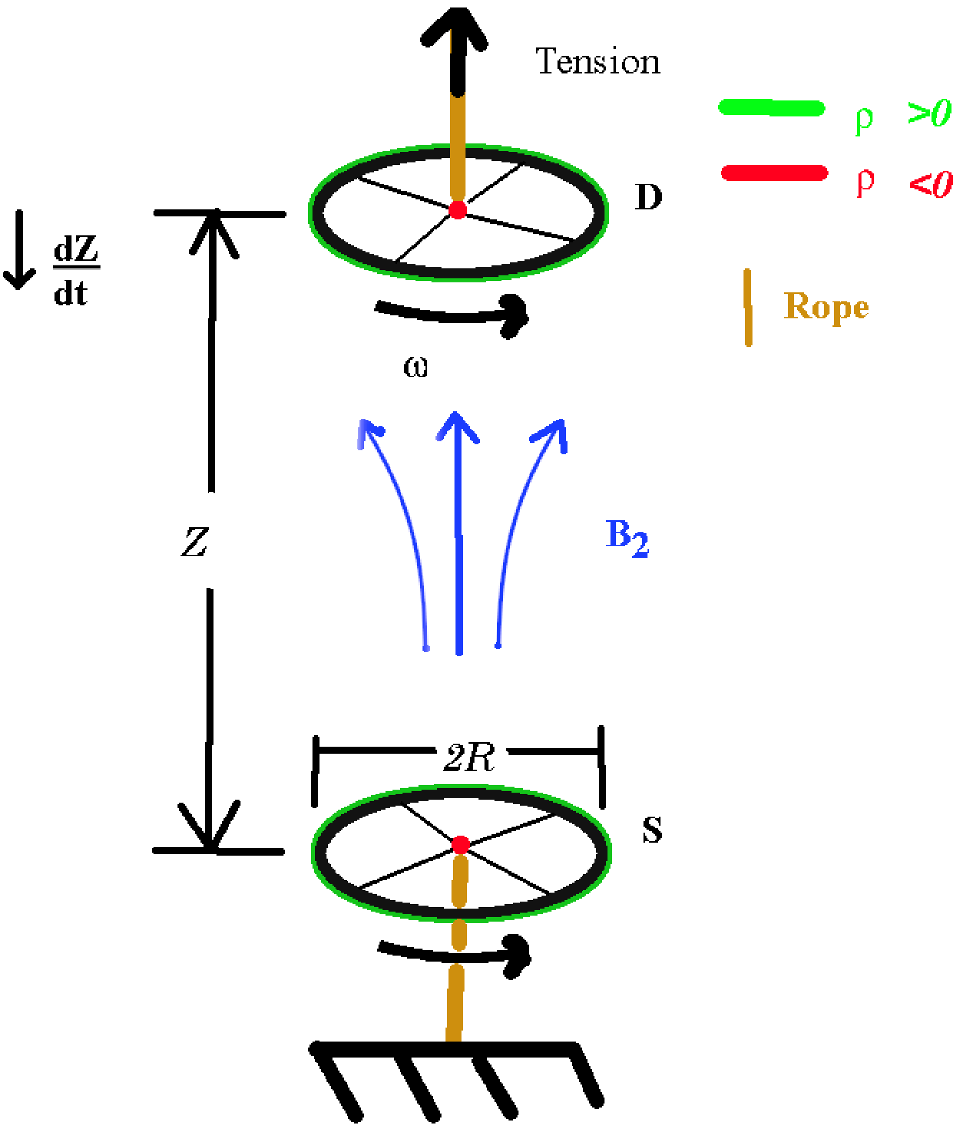

FIG. 1: Figure mag-dipole. Magnetic dipole model wheels

We note in each case that the rate of energy extraction goes as $\partial_t Z$ which can be made as small as desired. The effect is secular (additive) so by carrying out the process for a suffiently long time, one gets a non zero effect even as the velocity goes to zero. Ultimately it will be govrned by change in $\vec{A}$ not by the rate at which this change occurs. Of course, if the process is too fast, electromagnetic radiaiton will also be generated and the full Maxwell equations will be needed.

The bicycle wheel model is simply one model of many for the magnetic dipole (just as two point particles on the end of rigid struts is one of many models for the electric dipole). Given the model one must find the mechanical degrees of freedom which act as the energy reservoir for the magnetic diple. In a permant magnet, it could be the growth or the decay of the magnetic domains which could act as the mechanical reservoir. Or the angular momentum of the electrons in orbit about the nucleus could act like the reservoir. What is fascinating is that the magnetic dipole acts very differently to the electric dipole from the point of view of the detailed physics.

Not e also, a manrtra of mant texrbooks is that the magnetis fielf cannot do work. This is a misphrasing. As we see in the case of the upper D dipole. the magnetic field can do work. It soes work on both the wheel itself and on the external world. It is that the world cannot do work on the electromabnetic field via the magnetic field. It can do so only via the electric field via the $\vec{E}\cdot\vec{J}$ term in the conservation of Electromagnetic energy term.

The magnetic dipole problem is fascinating because it uses so many aspects of the Electromagnetic field equations.

$-E\cdot J$ Work done on the EM field by outside agents.(always true)
$\vec{E}=-\vec{n}abla\Phi-\partial_t\vec{A}$ Maxwell's eqn. (Always true)
$\vec{A}=\frac{\mu_0}{4\pi|x-x'|}\vec{J}$ (statics)
$\mathcal{E}_I=\rho_1\Phi_2+\vec{J}_1\cdot\vec{A}_2$ Interaction Energy of EM field (statics)
$\vec{F}=\rho\vec{E}+\vec{J}\times\vec{E}$ force between EM field and matter (always)
$\int\vec{J}d\mathfrak{V}=0$ Total current (statics)
$\int x^iJ^jd\mathfrak{V}=\frac{1}{2}\int(x^iJ^k-x^jJ^i)d\mathfrak{V}$ (statics)