Magnetostatics in Materials

Just as for electrostatics in a media, we approach the magnetostatics in the same, way, by averaging over the various material properties.

By the commutation of derivaties with averaging, we get

$$< \vec{B} >= \nabla \times < \vec{A} > \tag{1}$$

However the other static Maxwell's equation

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{2}$$

is more difficult just as it is in electrostatics. While the commutativity of derivatives with averaging, this is true for the averages just as for the exact Maxwell's equations. But it is not nearly as useful as one would expect. Each atom with spin will have microscopic currents which can fluctuate from one atom to the other. Again when we integrate

$$\int f(x-x')\vec{J}(x')_\alpha d^3x' \tag{3}$$

we can expand around the location of the atom $\vec{x}_\alpha$, to get

$$< J >= \sum_\alpha \int \left(f(\vec{x}-\vec{x}_\alpha)J_\alpha(x') - (x'^j - x_\alpha^j)\partial_{x^j} f(x^j - x_\alpha^j)) \right) d^3x' + \int f(\vec{x}-\vec{x}')J_f(x')d^3x' \tag{4}$$

where J_α varies ovve the scale of an atom, and is such that $\int J_\alpha d^3x'$ is varying over the scale the atom is zero, while J_f is slowly varying over the averaging scale and is called the free current. It is not zero because the averaging just encompases a part of the current. Now $\int J_\alpha^i(x')d^3x'$ is 0, but $\int J_f^i(x')d^3x'$ is just approximately $J_f^i(x)$ While the average of the current integrated over the whole space is 0, over just the tiny portion where $f(x-x')$ for given x is non-zero, the integral is not zero.

Again as with the electro-medium, often $< J_f >$ is zero, it is more common for it to be non-zero.

We get

$$\epsilon^{ijk}\partial_j < B_k >= \mu_0(< J_f > + \epsilon^{ijk}\partial_j < M_k > \tag{5}$$

$$\nabla \times < \vec{B} > (x) = \mu_0 < J_f > (x) + \mu_0 \nabla \times < \vec{M} > (\vec{x}) \tag{6}$$

Defining

$$< \vec{B} >= \mu_0(< \vec{H} > + < \vec{M} >) \tag{7}$$

we get

$$\nabla \times < \vec{H} >=< J_f > \tag{8}$$

Again linear materials have $< M >= \chi_m < H >$ and can be eitehr dimagnetic, paramagnetic or ferromagnetic. Ferromagnetic materials are such that their ground state in the absence of externa fields is one in which M is not zero, and in which the directions of the magnetic moments correlate over long (many microns) distances . Thus $< M >$ can be non-zero, even in the absence of an external field (these are ferromagnets, from the latin word for iron, which is one of the easiest materials to have such ferromagnetism.

While magnetic and electric materials are very different, and magnetism and electrical behaviour are very different, we note that the behaviour of the materials in matter is very similar. If we look at materials where the free charges and currents are zero, we have, for systems where the behaviour is linear

$$\vec{D} = \epsilon\vec{E} \;\; ; \;\; \vec{B} = \mu H \tag{9}$$

$$\nabla \times \vec{E} = 0 \;\; ; \;\; \nabla \times \vec{H} = 0 \tag{10}$$

$$\nabla \cdot \vec{D} = 0 \;\; ; \;\; \nabla \cdot \vec{B} = 0 \tag{11}$$

So beguiling has this symmetry looked that many wold argue that it is $\vec{E}$ and $\vec{H}$ that are Fundamental, and $\vec{D}$ and $\vec{B}$ that are secondary quantities, calling $\vec{H}$ the magnetic field, rather than $\vec{B}$. This is just silly. Yes,the approximate averaged equations would seem to have that parallelism, but it is really only true of linear materials. And it is B, not H that enters into the fundamental equaitons of the ineraction between matter and electromagnetism. Fortunately, in the absense of matter, B and H essentially equivaent.

Thus one can apply these equations to the magnetic eqivalent of the use of a high paramagnetic maerial as shielding.

Again having a material with a magnetic permittivity of μ, and a hollow sphere with inside radii $r1$ and $r2$ we get, assuming the equivalence $\vec{H} \leftrightarrow \vec{E}$, $\vec{B} \leftrightarrow \vec{D}$, $\mu \leftrightarrow \epsilon$

$$\frac{H_z inside = H_0(9\mu/\mu_0 r2^3}{9\mu/\mu_0 r2^3 + 2(\mu/\mu_0 - 1)^2 r2 - r1))(r2^2 + r1r2 + r1^2)} \tag{12}$$

which for large μ goes roughly as $\frac{9}{2\mu/\mu_0}H_0$. But for amaterial like mu-metal, it has μ/mu_0 of the order of 100,000, so it will decrease the magnetic field inside the spere by 10000 or so.

I. APPENDIX

Where does the equation for the relation between B and M come from? Again we expanded the averaging function $f(x-x')$ in a taylor series around the location of the atoms or molecules. In doing so we get tems like

$$< \vec{J^i} > (x) = \sum_\alpha \int \left(f(x-x_\alpha) J^i_\alpha(x'-x_\alpha) - \partial_i f(x-x_\alpha)(x'^j - x^j_\alpha) J^i(x'-x_\alpha) \right) d^3x' \tag{13}$$

$$= \sum_\alpha (-\frac{1}{2}[\partial_j f(x-x_\alpha) \int (x'^j - x^j_\alpha J^i - J^j(x'^i - x_\alpha)^i))d^3x' \tag{14}$$

In vector terms this is

$$< \vec{J} > (x) = -\sum_\alpha \int \frac{1}{2} \left((\nabla f(x-x_\alpha) \cdot (\vec{x}' - \vec{x}_\alpha)) \vec{J}(x') - \left(\nabla f(x-x_\alpha) \cdot \vec{J}(x') \right) (\vec{x}' - \vec{x}_\alpha) \right) d^3x' \tag{15}$$

Using the cross product identity

$$\vec{X} \times (\vec{Y} \times \vec{Z}) = (\vec{X} \cdot \vec{Z})\vec{Y} - (\vec{X} \cdot \vec{Y})\vec{Z} \tag{16}$$

we get

$$< \vec{J} >= \int \left(\nabla \sum_\alpha f(x-x_\alpha) \times ((\vec{x}' - \vec{x}_\alpha) \times -vec J_\alpha(x'-x_\alpha)) \right) \tag{17}$$

$$= \sum_\alpha \nabla f(x-x_\alpha) \times \vec{M}_\alpha) = \nabla \times < \vec{M} > (x) \tag{18}$$

as used above.
