Maxwell Equations

The fundamental dependent variables are a scalar field $\phi(t,\mathfrak{X})$ and a cotangent vector field $A_it,\mathfrak{X}$. Ie, these fields are functions that exist everywhere in space and time. Use these to define and Electric and a magnetic field

$$1 \;\; E_i = -\partial_i\phi - \partial_t A_i \tag{1}$$
$$2 \;\; B^i = \epsilon^{ijk}\partial_j A_k \tag{2}$$

From these definitions we get two identities

$$1' \;\; \epsilon^{ijk}\partial_j E_k = -\epsilon^{ijk}\partial_j\partial_k\phi - \partial_t B^i = -\partial_t B^i \tag{3}$$
$$2' \;\; \frac{1}{\sqrt{g}}\partial_i\sqrt{g}\epsilon^{ijk}\partial_j A_k = \epsilon^{ijk}\partial_i\partial_j A_k = 0 \tag{4}$$

Note that these depend on the metric only through the determinant of the metric.

The other two equations are

$$3 \;\; \frac{1}{\sqrt{g}}\partial_i g^{ij}E_j = \frac{1}{\epsilon_0}\rho \tag{5}$$
$$4 \;\; \epsilon^{ijk}\partial_j(g_{kl}B^l) = \mu_0\epsilon_0\partial_t g^{ij}E_j + \mu_0 J^i \tag{6}$$

where $\epsilon_0\mu_0 = \frac{1}{c^2}$ with c the velocity of light, and both c and μ_0 being defined, not measured, quantities. $\mu_0 = 4\pi 10^{-7}$. The arise because of the separate definition of the Volt (used in E) and the Coulomb used in defining both the Coulomb and current.

I. CONSERVATION OF CHARGE

Take the **div** of the 4th Maxwell equations:

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\epsilon^{ijk}\partial_j g_{kl}B^l = \mu_0\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}J^k) + \partial_t\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{kl}E_l \tag{7}$$

But the LHS is 0, because $\epsilon^{ijk} = \frac{1}{\sqrt{g}}e^{ijk}$ and that gives $\partial_i(\sqrt{g}\epsilon^{ijk}\partial_j g_{kl}B^l = e^{ijk}\partial_i\partial_j g_{kl}B^l = 0$ since the derivatives commute. Thus

$$0 = \mu_0(\frac{1}{\sqrt{g}}\partial_i J^i + \partial_t\rho) \tag{8}$$

Integrating over the a volume we have

$$\partial_t Q = \int \rho dV = -\int J^{\perp}dS. \tag{9}$$

Ie the rate of change in time for the total charge Q inside a volume equals the perpendicular flow of current into that volume through the surface. This equation is the conservation of charge equations. Note that this equations MUST be satisfied or Maxwell's equations are inconsistent.

II. GAUGE FREEDOM

If we consider Maxwell's equation, especially the definition of E_i and B^i in terms of ϕ, A_i, there is a freedom there. Consider the transformation

$$\hat{A}_i = A_i + \partial_i\psi \tag{10}$$
$$\hat{\phi} = \phi - \partial_t\psi \tag{11}$$

then the E and B, and thus the Maxwell equations in terms of E and B, are left unchanged. $\hat{E}_i = E_i$ and $\hat{B}^i = B^i$. This is called a gauge transformation. If one believes that only the electric and magnetic fields are measurable, then regarding ϕ and A_i as being fundamental seems a bit perverse. On the other hand, if one goes to a Lagrangian or Hamiltonian approach, then the only way one can write a Lagrangian for the electromagnetic field is by using ϕ and A_i. Furthermore, there are cases where one can actually measure aspects of A_i directly (Aharonov Bohm effect, which is not just a quantum effect, even though that was where it was discovered, but is also a classical effect.) It depends directly on the integral of A_i along a closed path. Since the vector potential depends non-locally on the B^i field, this means that a magnetic field can have effects on things which never enter the region in which the magnetic field is non-zero.

Gauge freedom is something like coordinate freedom. Any physical quantity must remain the same when the gauge is changed.

The gauge allows one to change the potentials. For example if we choose $\psi(t, \mathfrak{X} = \int \phi(t, \mathfrak{X}$, then the new scalar potential will be 0. If we chose $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\psi) = -\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}A_j)$ then the divergence of the new vector potential is 0. (Note that it is always possible to choose ψ in this way. This is called the Coulomb gauge. Or we could choose the gauge so that

$$\frac{1}{c^2}\partial_t\phi + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}A_j) = 0 \tag{12}$$

after the gauge transformation (again this is always possible, as long as the original ϕ, A_i fall off towards infinity sufficiently fast. This is called the Lorenz (after L. Lorenz) not to be confused with H. Lorentz of Lorentz transformations) gauge, and is most useful when looking at free electromagnetic waves. Again, it is only valid if the spatial metric is time independent.

Both B^i and E_i are invariant under any gauge transformations. This would mean that they are physical quanties which could be measured (ignoring the coordinate dependence). However there are also aspects of the potentials themselves which are also gauge invariant. For example if we take the integral of A_i around a closed curve ($\frac{\int_0^1 A_i(dx^i}{d\lambda d\lambda}$ where $x^i(\lambda = 0) = x^i(\lambda = 1)$, or a closed curve), then $\int \partial_i\psi\frac{dx^i}{d\lambda} = \psi(t, \mathfrak{X}(\lambda = 1)) - \psi(t, \mathfrak{X}(\lambda = 0)) = 0$ because all the coordinates are the same at $\lambda = 0$ and $\lambda = 1$. Thus the path integral of A_i is independent of the gauge, and could therefore be measurable. That they are indeed measurable took about 100 years for a graduate student, Y Aharonov (with help from his supervisor and from MHL Pryce), to notice in 1959. That it is measurable is now called the Aharonov Bohm effect (Bohm was Aharonov's PhD supervisor), and it actually has been measured. It was first discovered in quantum mechanics, but is also true for classical field theories.

We know that $B^i = \epsilon^{ijk}\partial_j A_k$ and thus by Stokes thm, the integral of a surface of the perpendicular component of B^i to the surface is equal to the integral of A_i around the boundary of the surface. Thus, if B^i is non-zero only in a limited region, A^i can be non-zero outside that region and one can do an experiment with charges which are always outside the region where B^i is non-zero to measure that gauge invariant path integral of A_i. So one must say either that aspects of A_i are real even in regions where there is no magnetic or electric fields, or one must say that the charges non-locally detect the magnetic fields. This latter is the attitude of many physicists, which seems perverse, when you can alternatively just say that the vector potential is physical and measurable.
