Physics 301
Midterm Exam
Feb 28 2021

This exam is **50 min** in length. It is closed book, but one 8.5x11 inch sheet of paper is allowed. No computers, phones, calculators are allowed

This exam has **5 questions**, all of equal value.

On your answer sheet, please include a statement that you have not and will not receive help in answering any of the question from any other person, and that you have not and will not read any material, whether in books or online to help you answer these questions, and sign that declaration. (Note one sheet of formulas is allowed).

When the exam has been marked, and handed back to you, you will have the option of redoing any of the questions on which you did not get full marks and handing the result in for remarking. As your midterm mark you will get the average of the mark you received on the midterm writing itself and the mark you get on the redo of the question, but in no case will you get less than that the mark you received on the midterm marking itself. This "re-do" will be due 1 week after the results of the marking of the original midterm are sent back to you, with no extension of that time. You may discuss this redo with others, and use any notes, text-books, etc. in doing so, except you may not simply copy someone else's solution(s), or feed them into any computer programs like, but not limited to ChatGPT.

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1. Which of the following are tensor equations. Explain why you say whether they are or not. Each individual lettered term on the RHS is assumed to be a tensor. ϵ_{ijk} is the completely anti-symmetric tensor such that $\epsilon_{123} = \sqrt{g}$. $g = det(g_{ij})$, $e^{ijk} = \sqrt{g}\epsilon^{ijk}$

a)

$$S_i{}^j = W_j Q^{kl} W_l + 5S^l (Q_{il} - Q_{li}) T^j$$

Yes. each term has the same free indices, the sum is over the a tangent and cotangent index.

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b)

$$S^i = \epsilon^{ijk} \partial_i A_k$$

Yes. This is just the expression of the curl of A. The possibly problematic terms of the derivative of the Jacobian matrices cancel out due to the antisymmetry.

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c)

$$S^i = e^{ijk} \partial_i A_k$$

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e^{ijk} is not a tensor, thus this expression is not. Since ϵ^{ijk} is a tensor, while the scalar, $\sqrt{g}$ is not a scalar function (it changes under a coordinate transformation) the product is not a matrix.

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d)

$$F = \partial_i A^i$$

No.

The tensor expression would be

$$\frac{1}{\sqrt{g}} \partial_i \sqrt{g} A^i$$

. The problem is that the transformation

$$\begin{aligned} &\partial_i \tilde{x}^j \partial \tilde{j} (\partial_{\tilde{k}} x^i \tilde{A}^k \\ &= \partial_i \tilde{x}^j \partial_{\tilde{k}} x^i \partial_{\tilde{j}} \tilde{A}^k + \partial_i \tilde{x}^j (\partial_{\tilde{j}} \partial_{\tilde{k}} x^i) \tilde{A}^k \end{aligned} \tag{1}$$

While the first termson the far right side is just what one would expect for a tensor, the second term, the second derivative of x^i is not. Ie, this does not transform as a tensor.

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e)

$$S_{ab} = W_a + R_b$$

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This term is not a tensor equation since

$$\tilde{S_{ab}} = \partial_{\tilde{a}} x^c \partial_{\tilde{b}} x^d S_{cd}$$

Ie the product of two Jacobian matrices, while the right side give

$$\partial_{\tilde{a}} x^c W_c + \partial_{\tilde{b}} x^d R_d$$

which does not have a product of the two Jaconbian matrices.

[1] for each line, (ie, 1/2 each for correctly getting whether it is a tensor relation, and 1/2 for reason

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[5] 2)The surfaces $x = 0, \ y = 0, \ z = 0$ are all conductors. A charge Q is placed at the point $x = y = z = 1$. Where are the image charges located and what are each of their values?

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The image charges are designed so as to make the potential along each conductor be a constant, and in this case, since the conductors go to infinity, zero. Thus the potential of the image charge must be equal and opposite at the surface of the conductor to the charge one is trying to cancel. Thus , to make the potential equal and opposite to the original charge along x=0. one would need a charge of -Q at x=-1, y=1, z=1. Now, to make the potential along y=0 we need to cancel the potentials of both the original charge and the image charge along y=0. Thus we need two image charges one at x=z=1 y=-1 of charge -Q (to cancel the potential of the original charge) and at z=1, y=x=-1 of charge +1 to cancel the first image charge at y=z=1 and x=-1. along y=0. This gives 4 in a square on the surface z=1. To now cancel the potentials along z=0. we need equal and opposite charges on $z = -1$. Thus one has the original charge and 7 image charges at the corners of the cube

$$(1,1,1)(+Q), \quad (1,1,-1)(-Q), \quad (1,-1,1)(-Q), \quad (-1,1,1)(-Q),$$
$$(1,-1,-1)(+Q), \quad (-1,1,-1)(+Q), \quad (-1,-1,1)(Q), \quad (-1,-1,-1)(-Q)$$

[1/3 for each location, and 1/3 for each sign of charge.]

. 3. We have a charge density distribution on the surface of a sphere of radius R given by

$$\rho(r, \theta, \phi) = \sigma \delta(r - R) \cos(\theta) \tag{2}$$

[2] a) What do you expect the θ, ϕ dependence of the potential for this charge distribution to be?

[3]b)What do you expect the radial dependence of the potential Φ to be both for $r < R$ and for $r > R$. (you do not need to solve the equation exactly– just give arguments as to what you expect the answer to be

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The charge distribution has an angular dependence of $cos(\theta)$ which is proportional to Y_{10}. Since this is a single spherical harmonic, the potential should have the same dependce for all r. so the angular dependence is cos(θ) Now there are two possible approaches. One is to substitute this into the potential equation

$$\frac{1}{r^2}\partial_r r^2 \partial_r (f(r)cos(\theta) \tag{3}$$

$$+\frac{f(r)}{r^2 sin(\theta)}\partial_\theta sin(\theta)\partial_\theta cos(\theta) + \frac{f(r)cos(\theta)}{r^2 sin(\theta)^2}\partial_\phi^2 1 \tag{4}$$

$$= -\sigma\delta(r-R) \tag{5}$$

$$\frac{1}{r^2}\partial_r r^2 \partial_r f(r) - \frac{2}{r^2} f(r) \tag{6}$$

$$= -\sigma\delta(r-R)\partial_r^2 f(r) + \frac{2}{r}\partial_r f(r) - \frac{2}{r} = -\sigma\delta(r-R) \tag{7}$$

Ie, it must solve the free equations for both $r > R$ and $r < R$. as done in class, this means that f(r) must be a linear combination or r and $1/r^2$, but with the first being the solution for $r < R$ and the second for $r > R$. or else the solution will diverge at $r = 0$ or $r = \infty$.

So outside it must go as α/r^2 and inside as βr. Now, it must be continuous at $r = R$ since otherwise that step function there would produce a first derivative which is a delta function, and the second would be the derivative of a delata function, and the equn says the second derivative is just the delta function. Thus $\alpha/R^2 = \beta R$, or $\alpha = \beta R^3$. Thus

$$f(r) = \beta R^3/r^2$$

for r¿R, and $f(r) = \beta r$ for $r < R$. The first derivative would be $-2\beta R^3/R^3$ for $r = R$ from the positive side, and equals β for $r = R$ from the negative sice. Ie, the difference is -3β. This step will produce a delta function of -3β. But the delta function is supposed to be $\frac{-\sigma}{\epsilon_0}$ so

$$\beta = \frac{\sigma}{3\epsilon_0}; \quad \alpha = \beta R^3 \tag{8}$$

This gives

$$f(r) = \frac{\sigma}{3\epsilon_0} r; \quad r < R; \quad \frac{\sigma R^3}{3r^2}; \quad r > R \tag{9}$$

Marks:2 for recognizing that the ongular dependence is just Y_{10}, and thus the field must be as well. 1 for recognising that the distant ($r > R$ must go as $1/r^2$, while for $r < 1$ as r. 1 for continuity of the potential at r=R and 1 for getting the junction correctly.

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[5] 4.A conductor has an interior electric field is equal to 0 because an electric field would push the charges around until the electric field is zero inside (but not on the surface). Show that the external electric field must be perpendicular to the surface just at the surface. What is the relation between the surface electric field and the surface charge density in the conductor?

The curl of the Electric field is 0, if we are in the static approximation (ie, the B field is independent of time). Thus if we create a path that runs along parallel to the conductor just outside the conductor, runs perpendicular to the conductor into the conductor for a timy way, and then run back paralle to the outside path, then Stoke's thm says that the integral of the component of E parallel to this path equals the integral of the curl of E over the surface that this path encloses. But the curl of E is zero. Also E along the path inside the conductor is zero. Thus the integral along the path just outside the conductor is also zero (the two end point contributions can be made arbitrarily small.) Ie, the component of E parallel to the conductor must be zero just outside the conductor. Since I can choose that path as short or as long as I want and pointing in any direction, which means that the component of E in all directions parallel to the conductor must be 0.

Similarly we can create a "pill box" whose top and bottom are parallel to the conductor whose top is just uutsie the conductor and whose bottom is just inside, separated by as small a distance as one wants. Then Gauss' thm says that the integral of the perpendicular component to the surface of this pill box equals the divergence of E inside the pill box. The E on the bottom surface is 0, so the integral of the perpendiculat component of E over the top of the pillbox, must equal the divergence of E inside the box. But the divergence of E is equal to the charge enclosed. Thus, the charge density per unit surface(which must lie on the surface of the conductor) must equal ϵ_0 times that perpendicular component of the E field.

[2] for using Stokes Thm to get parallel is zero, 1 perpendicular non-zero. 2 for relation between perp electrical field and surface charge density

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ℓ = 0

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

ℓ = 1

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x-iy)}{r}$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\frac{z}{r}$$

$$Y_1^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x+iy)}{r}$$

ℓ = 2

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)^2}{r^2}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)\cdot z}{r^2}$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot\frac{(3z^2-r^2)}{r^2}$$

$$Y_2^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)\cdot z}{r^2}$$

$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)^2}{r^2}$$

ℓ = 3

$$Y_3^{-3}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{35}{\pi}}\cdot e^{-3i\varphi}\cdot\sin^3\theta = \frac{1}{8}\sqrt{\frac{35}{\pi}}\cdot\frac{(x-iy)^3}{r^3}$$

$$Y_3^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta\cdot\cos\theta = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot\frac{(x-iy)^2\cdot z}{r^3}$$

$$Y_3^{-1}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{21}{\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot(5\cos^2\theta-1) = \frac{1}{8}\sqrt{\frac{21}{\pi}}\cdot\frac{(x-iy)\cdot(5z^2-r^2)}{r^3}$$

$$Y_3^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{7}{\pi}}\cdot(5\cos^3\theta-3\cos\theta) = \frac{1}{4}\sqrt{\frac{7}{\pi}}\cdot\frac{(5z^3-3zr^2)}{r^3}$$

$$Y_3^{1}(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{21}{\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot(5\cos^2\theta-1) = \frac{-1}{8}\sqrt{\frac{21}{\pi}}\cdot\frac{(x+iy)\cdot(5z^2-r^2)}{r^3}$$

$$Y_3^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta\cdot\cos\theta = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot\frac{(x+iy)^2\cdot z}{r^3}$$

$$Y_3^{3}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{35}{\pi}}\cdot e^{3i\varphi}\cdot\sin^3\theta = \frac{-1}{8}\sqrt{\frac{35}{\pi}}\cdot\frac{(x+iy)^3}{r^3}$$

Figure 1: Figure ylm. From https://en.wikipedia.org/wiki/Table_of_spherical_harmonics with signs altered to make $Y_{lm}^* = Y_{l,-m}$.