Potential, delta function, and Green's function

We are going to be interested in the static limit of Maxwell equations, ie where the time derivative is negligible, and can be approximated by zero. This is clearly an approximation, since anything we do or measure has time dependence, but often it is a reasonable approximation.

Maxwell's equations then reduce to (Using Φ for the potential so as not to confuse it with the coordinate ϕ).

$$1 \quad E_i = -\partial_i \Phi \tag{1}$$
$$1' \quad \epsilon^{ijk}\partial_j E_j = 0 \tag{2}$$
$$2 \quad B^i = \epsilon^{ijk}\partial_j A_k \tag{3}$$
$$2' \quad \frac{1}{\sqrt{g}}\partial_i \sqrt{g} B^i = 0 \tag{4}$$
$$3 \quad \frac{1}{\sqrt{g}}\partial_i \sqrt{g} g^{ij} E_j = \frac{1}{\epsilon_0}\rho(t, \mathfrak{X}) \tag{5}$$
$$4 \quad \epsilon^{ijk}\partial_j g_{kl} B^l = \mu_0 J^i(t, \mathfrak{X}) \tag{6}$$

Note that the equations split up into equations for the electric fields and for the magnetic fields separately, whic means that in the static limit, the theory splits into two separate areas, electric and magnetic.

eqnds 1' and 3 give

$$\frac{1}{\sqrt{g}}\partial_i \sqrt{g} g^{ij}\partial_j \Phi = \rho(\mathfrak{X}) \tag{7}$$

To begin let us first look at this equation in spherical polar coordinates.

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \tag{8}$$

with coordinates $x^1 = r; \quad x^2 = \theta; \quad x^3 = \phi$. Then

$$g_{11} \equiv g_{rr} = 1; \quad g_{22} \equiv g_{\theta\theta} = r^2; \quad g^{33} \equiv g_{\phi\phi} = r^2 sin(\theta)^2 \tag{9}$$

and all other components 0. Thus the metric matrix is diagonal and the determinant, g is $r^4 \sin(\theta)^2$

Thus the equation for Φ is

$$\frac{1}{r^2\sin(\theta)}\left(\partial_r r^2 sin(\theta) 1 \partial_r \Phi + \partial_\theta r^2 sin(\theta)\frac{1}{r^2}\partial_\theta \Phi + \partial_\phi r^2 \sin(\theta)\frac{1}{r^2\sin(\theta)^2}\partial_\phi\right)\Phi = -\frac{1}{\epsilon_0}\rho \tag{10}$$

Let us firstly assume that we have no θ or ϕ dependence in Φ, so the only derivative that survives is the r derivatives. This leads to

$$\frac{1}{r^2}\partial_r r^2 \partial_r \Phi(r) = -\frac{1}{\epsilon_0}\rho \tag{11}$$

Let us first assume that $\rho = 0$. Then multipy by r^2, integrate by r to get

$$r^2 \partial_r \Phi(r) = C \quad \text{a constant} \tag{12}$$

Dividing by r^2 and integrating, we get $\Phi(r) = -\frac{C}{r} + D$, another constant.

This solution has some weird features. The volume integral factor is

$$dV = \sqrt{g} dx^1 dx^2 dx^3 = r^2 \sin(\theta) dr d\theta d\phi \tag{13}$$

Using Gauss thm and recalling that the r component of E is perpendicular to the surfrace r=R const, we get

$$E_r = -\partial_r \Phi = -\frac{C}{r^2} \tag{14}$$
$$\int g^{rr} E_r R^2 \sin(\theta) d\theta d\phi = \int \frac{\rho}{\epsilon_0} r^2 sin(\theta) dr d\theta d\phi \tag{15}$$

But $g^{rr} E_r R^2 sin(\theta) d\theta d\phi = 4\pi R^2 \frac{-C}{R^2} = -4\pi C$. This tells us that the integral of ρ must be non zero. But this equation is true no matter what the value of R is, including arbitrarily small. Thus the charge must be located at r=0, and the integral of ρ/ϵ_0must be non-zero and have value $-4\pi C$.

The charge density must have the property that it is non-zero at only a single point, and and its integral over the volume must be $-4\pi C$. But a fundamental property of integrals is that the integral over any function cannot depend on its value at a single point. This "thing" is called a distribtion, and is something which is zero everywhere except at a point, but its integral is finite. Mathematicians call this a distribution. It is something that is not a function, but something whose integral is non-zero. Since we are working in 3 dimensions, this is called a three dimensional delta function $4\pi C \delta^3(r)$. Ie, the δ^3 function is something which when integrated over the volume containing point where the argument of δ^3 is zero gives the value 1. The δ^3 thus has dimensions of $1/distance^3$). There is clearly nothing physical that could have these properties.

A one dimensional delta function is a distribution such that it 0 everywhere except at $x = 0$ but the integral over it is unity. It is called $\delta(x)$. nd example is $\frac{d\Theta(x)}{x}$ where $\Theta(x)$ is the Heaviside step function such that $\Theta(x) = 1$ if x¿0 and is 0 if x¡0. If we have a regular function $f(x)$ then

$$\int -a^a f(x) \frac{d\Theta(x)}{dx} dx$$

is independent of a. But integrating by parts we have

$$\int_{-a}^{a} f(x) \frac{d\Theta(x)}{dx} dx = \int_{-a}^{a} \frac{d}{dx}(f(x)\Theta(x)) - \int_{-a}^{a} \frac{df}{dx} \Theta(x) dx$$
$$= f(a) - \int_{0}^{a} \frac{df(x)}{dx} dx = f(a) - (f(a) - f(0)) = f(0) \quad (16)$$

Ie, the integral over any regular function times the delta function simply gives the value of the function at $x = 0$.

If one wants the delta function at any other point, we simply write $\delta(x - x') = \frac{d\Theta(x-x')}{dx}$

Let us change coordinates in the potential equation above.

$$z = rcos(\theta); \quad x = r\sin(\theta)\cos(\phi); \quad y = rsin(\theta)sin(\phi) \quad (17)$$

Since $\Phi(r, \theta, \phi) = \Phi(r(x, y, z), \theta(x, y, z), \phi(x, y, z)$ as it is a scalar, and as the equation is a tensor equation. The metric becomes

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (18)$$

and $r = \sqrt(x^2 + y^2 + z^2)$. The eqn 1 and 3 become

$$\partial_x^2 \Phi + \partial_y^2 \Phi + \partial_z^2 \Phi = \frac{-\rho(x, y, z)}{\epsilon_0} \quad (19)$$

and $\Phi = -\frac{1}{4\pi r} = -\frac{1}{4\pi\sqrt{x^2+y^2+z^2}}$

$$\partial_x^2 \Phi + \partial_y^2 + \partial_z^2 \Phi = -\delta(x)\delta(y)\delta(z) \quad (20)$$

Or if one centers r at other points,

$$\Phi = -\frac{1}{4\pi r} = -\frac{1}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \quad (21)$$
$$\partial_x^2 \Phi + \partial_y^2 + \partial_z^2 \Phi = \delta(x - x')\delta(y - y')\delta(z - z') \quad (22)$$

Maxwell's equations are linear, so if we add two solution, including their charge densities, we still get a solution with charge density being the sum of the charge densities. In this case we have a solution whose charge density is the all located at a single point. By adjusting the amplitude of that at any point and summing over all points, one can get a solution with arbitrary charge density. Note also that integrals are just summations.

Thus if we do

$$\tilde{\Phi}(x, y, z) = \int \frac{1}{\epsilon_0 4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} (-\rho(x', y', z')) dx'dy'dz' \quad (23)$$

then $\tilde{\Phi}(x,y,z)$ obeys

$$\partial_x^2\tilde{\Phi}(x,y,z)+\partial_y^2\tilde{\Phi}(x,y,z)+\partial_z^2\tilde{\Phi}(x,y,z)=\delta(x-x')\delta(y-y')\delta(z-z')\frac{-\rho(x',y',z')}{\epsilon_0}dx'dy'dz'=-\frac{\rho(x,y,z)}{\epsilon_0} \tag{24}$$

Ie, we can solve the equation by use of what is called a Green's fution integrated over the RHS of the equations. Ie, a devivative equaiton is turned into an integral equation. All linear equations with source have this behaviour. One can find a function, such that summing or integrating it over the source terms gives one a solution to the equation with the specific source.

Ie, we have

$$G(x,y,z;x',y',z')=\frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{25}$$

There are actually two Green's functions. The first is the standard one, namely the one correspoinding to

$$(\partial_x^2G(x,y,z;x',y',z')+\partial_y^2G(x,y,z;x',y',z')+\partial_z(x,y,z;x',y',z'))=\delta^3(\mathfrak{X}-\mathfrak{X}')=\delta(x-x')\delta(y-y')\delta(x-x') \tag{26}$$

The other is the charge density Green's function, where

$$\begin{aligned}(\partial_x^2G_\rho(x,y,z;x',y',z')+\partial_y^2G_\rho(x,y,z;x',y',z')+\partial_zG_\rho(x,y,z;x',y',z'))\\ =-\frac{1}{\epsilon_0}\delta^3(\mathfrak{X}-\mathfrak{X}')=-\frac{1}{\epsilon_0}\delta(x-x')\delta(y-y')\delta(x-x')\end{aligned} \tag{27}$$

which corresponds to the potential of a delta funtion charge density. They are clearly related by

$$G_\rho(x,y,z;x',y',x')=-\frac{1}{\epsilon_0}G(x,y,z;x',y',z')$$

G_ρ is clearly the most useful if one is given a charge density, while G is the more usual mathematical definition of the Green's function, relating the potential to the inhomogeneous source term.

For the solutions to for charges in free space, we have for a source at $\mathfrak{X}'$

$$G_\rho=\frac{1}{4\pi\epsilon_0\sqrt{(x-x')^2+(y-y')^2(z-z')^2}} \tag{28}$$

I. MIRROR IMAGES

Consider a point charge outside a flat conducting wall, at potential equals 0. If the charge were located at $\mathfrak{X}'$ then the potential would be

$$\Phi(\mathfrak{X})=QG_\rho(\mathfrak{X},\mathfrak{X}') \tag{29}$$

where Q is the charge. But that conductive sheet alters the potential and the electric field of that charge. Let us assume that the sheet has potential equal to zero. That means that Φ must be equal to 0 along the sheet. (It is constant inside, because of the E field is 0 inside), and if the sheet has no net charge, it will be potential equal to 0. It must be continuous or the E field, which is the derivative of the potential, would have to have a δ function like distribution at the sheet. This would be bad, since, as we will discover later, the energy density in an electric field distribution goes as $\vec{E}\cdot\vec{E}$, would have a behaviour as δ^2, which, even when integrated, is infinite.

In order to be the field of the charge, the field must have the above form, but can also have added to that solution of the zero charge equations. Thus it will be

$$\Phi(\mathfrak{X})=G_\rho(\mathfrak{X},\mathfrak{x}')+F(\mathfrak{x},x')$$

where F obeys $\nabla^2F=0$ In order to solve the equation for the charge near a conducting wall, the potential on the wall must be 0. so we need $G(\mathfrak{X},\mathfrak{X}')+F(\mathfrak{X},\mathfrak{X}')|_{\mathfrak{S}}=0$ where $\mathfrak{S}$ means eveluated so that $\mathfrak{X}$ is a coordinate on the wall. Now this would seem to be hard to do, since we have to find an F such that its potential on the wall equals that of the G.But is is actually easy. If we place a charge on the exact opposite side of the wall, the value of the potential at the wall would be the same, if the charge were the same (the wall is a surface of symmetry), and its negative if the charge were opposite, thus if in flat space one were to place an equal but negative charge on the oppositee side of where the wall would be, then the potential along the wall would be zero. Also the potential field on the side of the wall as the original charge would solve the zero density equations, the Poisson equation, everwhere on that same side of the wall. Ie this would be a solution we want.

Thus, working Cartesian coordinates, with the wall at $x = 0$ we would have

$$\begin{aligned} \Phi(\mathfrak{X}) &= QG_\rho(\mathfrak{X}, \mathfrak{X}') - QG_\rho(\mathfrak{X}, \underset{x' \to -x'}{\mathfrak{X}'}) \\ &= Q\left(\frac{1}{\epsilon_0((x-x')^2 + (y-y')^2 + (z-z')^2)} - \frac{1}{\epsilon_0((x+x')^2 + (y-y')^2 + (z-z')^2)}\right) \end{aligned} \tag{30}$$

Now, this field looks as though it has two charges in it, but the second charge is "behind the conductor" which is the area we are not interested in. That second charge is not real. It's sole purpose is to provide a function F which obeys the source free Poisson equation in the region $x > 0$ and which cancels out the boundary value of G_ρ on the boundary. Because $\Phi(x = 0) = 0$, the E field parallel to the boundry would be zero because the derivatives of Φ along the y and z directions would be zero, since the derivative of a constant is 0.

This method or "mirror" charges, is one that can be applied to slightly more complicated situations. Lets say that we have not one conduction wall, but two infinite conducting walls. We place a charge between them. we not have a mirror charge beyond the conductor on the right. Both the original charge and that first mirror charge would also have mirrors charges in the left conductor. But both of those would then have mirror charges in the right conductor, with now there being 3 mirror charges in the right. The 2 new ones would have additional mirrors in the left, etc. Ie, you would get an infinite tower of image charges of opposite signs. Byt each new mirror set would be futher and further away from the region between the mirros and the effect would get weaker and weaker (1/(x-x')) would get smaller and smaller

This is like sitting in a hair cutters chair with mirrors on opposite walls in front of and behind you. You see this infinite ladder of images of yourself in the mirror. If the walls wee not quite parallel you would again get an infinite ladder of images, which instead of being lined up ahead of you would spreas out in a slanted line in front of and behind you. Ie, the solution of the potentials would be an infinte, converging sum of these mirror charges.

An interesing situation would occur if you had two conductors are right angles to each other. with the charge being in the enclosed wedge. In this cae you would have not only mirror charges, but mirror conductors. The solution would be what you would get as if there really were two intersecting infinite wall conductors. You would get 4 image charges. The first would be the relections from the $x = 0, y > 0$ conductor, of opposite sign, which would then be reflected from the mirror $y = 0, x < 0$ mirror conductor, with the same sign as the original, which would be reflected from the mirror $x = 0, y < 0$, to give a total of 4 charges, 3 of them image charges. conductor, which would be relected from the $y = 0, x < 0$ image conductor, to give a total of 4 charges, 3 of them image charges, two positive and two negative diagonally opposite the other one of the same sign.

The above is what would happen if the conductors were flat. The other situation where one gets a "simple" single mirror charge is if the conductor were spherical, with radius R. Since on has rotation symmetry, the image charge, if it exists, would have to like on the same line joining the the center of the spere to the the original charge. If the charge is ourside of the sphere, the mirror charge would be inside. It would have to be a negative charge, since the potential of a charge (G_ρ) is everywhere positive if the charge is positive, so the mirror charge would have to be negative to cancel the positive potential on the surface of the conductor from QG_ρ. Now let us assume that the charge is at $r = r_0 > R$ outside a spherical conductor. Then the mirror charge will locate at $r_1 = \frac{R^2}{r_0}$ and will have charge of $-QR^2/r_0$.

If the charge is outside the conductor ($r_0 > R$), then $r_1 < R$, and the mirror charge will be inside the sphere and its charge will be less than (and opposite sign to) the outsice charge.

If the system is a charge insice a spherical cavity, then the mirror charge will be outside the conductor, with the same relation as before.

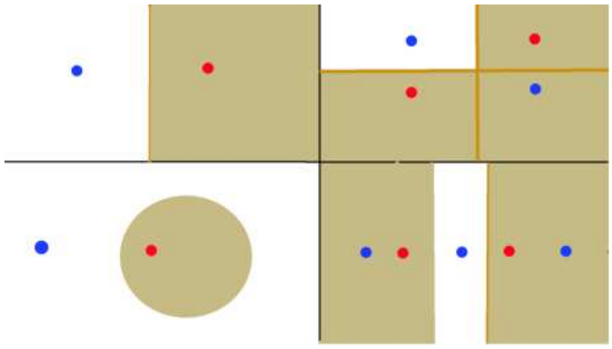

FIG. 1: Figure mirror1. Various images forcharges and mirror images. Blue=positiv, red=negative