Rodriquez Formula

Consider the formula for the angular momentum functions

$$\frac{1}{\sin(\theta)}\partial_\theta(sin(\theta)\partial_\theta P_{lm}(\theta)) + \frac{m^2}{sin(\theta)^2}P_{lm} = -\lambda P_{lm} \tag{1}$$

and let us first look at $m = 0$.

The first thing is to do a change of variables, setting $x = cos(\theta)$. we have

$$\partial_\theta = \partial_\theta x \partial_x = sin(\theta)\partial_x$$

which transforms the equation to

$$\partial_x\left((1-x^2)\partial_x\tilde{P}_\lambda(x)\right) = -\lambda\tilde{P}_\lambda(x) \tag{2}$$

where

$$P_\lambda(\theta) = \tilde{P}_\lambda(cos(\theta))$$

The first thing is that the $P_\lambda(\theta)$ are orthogonal for different λ.

$$-\lambda\int P_{\lambda'}(\theta)P_{\lambda'}(\theta)\sin(\theta)d\theta d\phi = \int P_{\lambda'}(\theta)\frac{1}{\sin(\theta)}\partial_\theta(sin(\theta)\partial_\theta P_\lambda)sin(\theta)d\theta d\phi \tag{3}$$

$$= -\int\partial_\theta P_{\lambda'}sin(\theta)\partial_\theta P_\lambda(\theta)d\theta d\phi = \int P_\lambda(\theta)\partial_\theta(sin(\theta)\partial_\theta P_{\lambda'}(\theta))d\theta d\phi = -\lambda'\int P_{\lambda'}(\theta)P_{\lambda'}(\theta)\sin(\theta)d\theta d\phi \tag{4}$$

This can be true only if $\lambda' = \lambda$ or if $\int P_{\lambda'}(\theta)P_{\lambda'}(\theta)\sin(\theta)d\theta d\phi = 0$

Note that this means that

$$\int_{-1}^{1}\tilde{P}_\lambda(x)\tilde{P}_{\lambda'}(x)dx = 0 \tag{5}$$

if $\lambda \neq \lambda'$

We can now look at the power series expansion of $\tilde{P}_\lambda(x)$

$$P_\lambda(x) = \sum_n a_n x^n \tag{6}$$

using the equation we find, setting each power of x to zero

$$a_{n+2}(n+2)(n+1) - a_n(n)(n+1) + \lambda a_n = 0 \tag{7}$$

For large n, this becomes approximately

$$a_{n+2} = a_n(\frac{n}{n+2} - \frac{\lambda}{(n+2)(n+1)}) \approx a_n(1-2/n) \tag{8}$$

$$\frac{da_n}{dn} \approx -1/na_n \tag{9}$$

$$a_n \approx -\frac{2R}{n}a_R \tag{10}$$

$$a_n x^n \approx \frac{2R}{n}a_{2R}x^{2R}x^{n-2R} \tag{11}$$

If x¿1, this means that for large enough n, each term in the series gets larger and larger and the series diverges. Ie, the solution goes singular at $x = \pm 1$. The only way to avoid this is to have a_n being zero for some value of n. This will occur for $n(n+1) = \lambda$ for some value of n.

Thus, the solutions must have $\lambda = l(l+1)$ to be regular at $x = \pm 1$

Thus the solution must be a polynomial in x with terms going from x^0 to x^l. There are l values of $\tilde{P}_l$, all orthogonal to each other under the integral from $-1to1$, so if we take linear sums, we can generate l polynomials with arbitrary coefficients for the various powers of x. And $\tilde{P}_{l+1}$ must be orthogonal to every polynomial of degree of l. Consider

$$\tilde{P}(x) = \partial_x^{l+1}(1-x^2)^{l+1} \tag{12}$$

It is a polynomial of degree $l+1$, and if Q(x) is a polynomial of degree l we want

$$\int_{-1}^{1} \partial_x^{l+1}(1-x^2)^{l+1} Q(x) dx = 0 \tag{13}$$

$\partial_x^l(1-x^2)^{l+1}$ must have at least a factor of $(1-x^2)$ since the l derivatives can hit at most l of the factors of $1-x^2$ at lest once.

Then

$$\int_{-1}^{1} \partial_x^{l+1}(1-x^2)^{l+1} Q(x) dx = -\int_{-1}^{1} \partial_x(\partial_x^l(1-x^2)^{l+1}Q(x)) - \int_{-1}^{1} \partial_x^l(1-x^2)^{l+1}\partial_x Q(x) dx \tag{14}$$

$$0 - \int_{-1}^{1} \partial_x^l(1-x^2)^{l+1}\partial_x Q(x) dx \tag{15}$$

since the factor $(1-x^2)$ in $\partial_x^l(1-x^2)^{l+1}$ is zero at both x=1 and x=-1. Doing integration by parts $l+1$ times gives us

$$\int_{0}^{1} (1-x^2)^{l+1}\partial_x^{l+1} Q(x) dx = 0 \tag{16}$$

since taking $l+1$ derivatives of a polynomial of degree l is zero.

Ie, if $\tilde{P}_{l+1}(x) \propto \partial_x^{l+1}(1-x^2)$ then if is orthogonal to all lower index $P_{r\leq l}(x)$ (and if we use the same expression, all higher ones will also be orthogonal.)

The standard expression is to choose $(P_l(1) = 1$. If two derivatives hit any one of the factors of $1-x^2$, then the rest of the expression must have a factor of $1-x^2$ left over that that is equal to 0 at $x = \pm 1$ each factor of $(1-x^2)^l$ Thus the only term which has a non-zero value is where all of the derivatives hit one of the factors once. That give

$$\partial_x^l(1-x^2)^l|_{x=1} = l!(\partial_x(1-x^2))^l + (1-x^2)(\text{some polynomial deg l} - 2)|_{x=1} = 2!(-1)^l + 0 \tag{17}$$

Thus the standard scaling is

$$\tilde{P}_l = \frac{1}{2^l l!}\partial_x^l(x^2-1)^l \tag{18}$$

These are called the Legendre polynomials.

The normalisation is

$$\int_{-1}^{1}\int_{0}^{2\pi} (\tilde{P}_l(x)^2 d\phi dx = (\frac{1}{2^l l!})^2 2\pi \int_{-1}^{1} \partial_x^l(x^2-1)^l \partial_x^l(x^2-1)^l dx \tag{19}$$

Doing l integrations by parts, we finally get

$$(\frac{1}{2^l l!})^2 \int_{-1}^{1} (x^2-1)\partial_x^{2l}(x^2-1)^l = (\frac{1}{2^l l!})^2 2^{2l}(2^l)! \int_{-} 1^1 (x^2-1)^l dx // = \frac{2}{2l+1} \tag{20}$$

These then give the solutions for the $m = 0$.

The solutions for $0 \neq 0$ are given by

$$\tilde{P}_{lm} = (-1)^m(1-x^2)^{|m|/2}\partial_x^| m|\tilde{P}_l(x) \tag{21}$$

where $\sqrt{1-x^2} = sin(\theta)$ and again $x = cos(\theta)$ Note that $|m|$ cannot be greater than l, since the $P_l(x)$ is a polynomial of degree 2l, and thus more than 2l derivatives would equal 0.

The normalisation is

$$\int\int \tilde{P}_{lm}(x)^2 dx d\phi = \frac{4\pi(l+|m|)!}{(2l+1)(l-|m|)!} \tag{22}$$

Then

$$Y_{lm}(x\phi) = \sqrt{\frac{(2l+1))(l-|m|)!}{4\pi(l+|m|)!}}\tilde{P}_{lm}(x)e^{im\phi} \tag{23}$$

Note that the P_{lm} are not orthogonal to each other, except for the same $|m|$ and different l.

I. RAISING OPEATOR

The operator

$$L_+ = L_y - iL_x = -(x+iy)\partial_z + z(\partial_x + i\partial_y) \tag{24}$$

is a raising operator the angular momentum. This operators equals

$$L_+ = -e^{i\phi}\left(\partial_\theta + i\frac{\cos(\theta)}{\sin(\theta)}\partial_\phi\right) \tag{25}$$

commutes with the angular operator

$$\frac{1}{\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta\partial_\theta + \frac{1}{\sin(\theta)^2}\partial_\phi^2 \tag{26}$$

and thus leaves $l(l+1)$ alone. The function $Y_{l,-l}(\theta,\phi) = N_{l,-l}e^{-il\phi}sin(\theta)^{2l}$ is a solution to the angular equation with $m = -l$, where $N_{l,-l}$ is a normalisation factor.

$$\frac{1}{\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta\partial_\theta(e^{-il\phi}sin(\theta)^l)) + \frac{1}{\sin(\theta)^2}\partial_\phi^2 e^{-il\phi}sin(\theta)^l = (l(l+1))e^{-il\phi}sin(\theta)^2 \tag{27}$$

If we operate on this function n times with the raising operator we get

$$Y_{l,-l+n}(\theta,\phi) = N_{l,-l+n}(-1^n)e^{i-l+n\phi}sin(\theta)^{-l+n}(\frac{1}{\sin(\theta)}\partial_\theta)^{-l+n}sin(\theta)^{2l} \tag{28}$$

Writing this with respect to $\xi = cos(\theta)$ and setting $-l+n = m$ we get Rodriques formula

$$Y_{l,m}(\xi,\phi) = N_{l,m}e^{im\phi}|1-\xi^{2m/2}|\partial_\xi^{l+m}(1-\xi^2)^l \tag{29}$$

since $\frac{1}{\sin(\theta)}\partial_\theta = -\partial_\xi$. That factor can be absorbed into N_{lm} or we can demand that $m = 0$ has a + sign, and then that sign factor becomes $(-1)^m$.
