Physics 530-23
Tutorial 1

1. Consider a tensor in 3-D and a coordinate transformation (where

$$\{x^1, x^2, x^3\} \equiv \{x, y, z\}$$

. Consider the coordinate transformation

$$\tilde{x} = x\cos(\theta) + y\sin(\theta); \quad \tilde{y} = y\cos(\theta) - x\sin(\theta); \quad \tilde{z} = z$$

where θ is a constant.Consider the following tensors

$$T^i = \{1, 2, 1\} \tag{1}$$
$$W_i = \{2, 1, 2\} \tag{2}$$
$$H^i{}_j : \quad H^1{}_1 = 1; \quad H^2{}_1 = -1 \tag{3}$$
$$\text{all others components } 0 \tag{4}$$

What are the following components in the tilde coordinate system.

$$\tilde{T^1} \qquad \tilde{W_2} \qquad \tilde{H^2}_2$$

The key here is to get the conversion from one coordinate system to the other . In this case the change is simply a rotation around the z axis of the xyz system

$$\tilde{x} = xcos(\theta) + y\sin(\theta); \quad tildey = ycos(\theta) - x\sin(theta); \quad \tilde{z} = z \tag{5}$$

Then the inverse transformation is

$$x = \tilde{x}\cos(\theta) - \tilde{y}\sin(\theta); \quad y = \tilde{y}cos(\theta) + \tilde{x}\sin(\theta); \quad z = \tilde{z} \tag{6}$$

Then

$$\partial_1\tilde{x}^1 = cos(\theta); \quad \partial_1 x^2 = \sin(\theta); \quad \partial_1 x^3 = 0 \tag{7}$$
$$\partial_2\tilde{x}^1 = -sin(\theta); \quad \partial_2\tilde{x}^2 = \cos(\theta); \quad \partial_2 x^3 = 0 \tag{8}$$
$$\partial_3\tilde{x}^1 = \partial_3\tilde{x}^2 = 0; \quad \partial_3 x^3 = 0; \tag{9}$$

and

$$\partial_{\tilde{1}} x^1 = cos(\theta); \quad \partial_{\tilde{1}} x^2 = -\sin(\theta); \quad \partial_{\tilde{1}} x^3 = 0 \tag{10}$$
$$\partial_{\tilde{2}} x^1 = sin(\theta); \quad \partial_{\tilde{2}}\tilde{x}^2 = \cos(\theta); \quad \partial_{\tilde{2}} x^3 = 0 \tag{11}$$
$$\partial_{\tilde{3}} x^1 = \partial_{\tilde{3}} x^2 = 0; \quad \partial_{\tilde{3}} x^3 = 0; \tag{12}$$

Then $\tilde{T}^i = \partial_j\tilde{x}^i T^j = \sum_j \partial_j\tilde{x}^i T^j$ or

$$\begin{aligned}\tilde{T}^1 &= \partial_1\tilde{x}^1 T^1 + \partial_2\tilde{x}^1 T^2 + \partial_3\tilde{x}^1 T^3 = \\ &= \partial\tilde{x}/\partial x T^x + (\partial\tilde{x}/\partial y)T^y + (\partial\tilde{x}/\partial z)T^z \\ &= cos(\theta)1 + (-sin(\theta))2 + 0\end{aligned} \tag{13}$$

Note that for the transformation of the tangent vector components, if the left side is the new tilde side, than the tilde coordinates are on top on the right hand side.

For W_i we have

$$\tilde{W}_2 = (\partial_{\tilde{2}} x^1)W_1 + (\partial_{\tilde{2}} x^2)W_2 + (\partial_{\tilde{2}} x^3)W_3$$

$$(\partial x/\partial \tilde{y})W_x + (\partial y/\partial \tilde{y})W_y + (\partial z/\partial \tilde{y})W_z$$
$$= cos(\theta)2 + sin(\theta)1 + 0 \tag{14}$$
$$\tag{15}$$

For $H^i{}_j$ it is a bit more complicated. We have

$$\tilde{H}^k_l = (\partial_i \tilde{x}^k)(\partial_{\tilde{l}} x^j H^i{}_j$$

rememnbering to sum over the repeated i and j. The only to non-zero terms have i=1,2 and j=1 so the only derivatives will be

$$\partial_1 \tilde{x}^k, \quad \partial_2 \tilde{x}^k$$

for k=1,2, and

$$\partial_{\tilde{l}} x^1$$

for $l = 1, 2.$ or k=1:

$$\partial_x \tilde{x} = \cos(\theta); \quad \partial_y \tilde{x} = sin(\theta)$$

k=2:

$$\partial_x \tilde{y} = -sin(\theta; \quad \partial_y \tilde{y} = cos(\theta)$$

and for l=1,2:

$$\partial_{\tilde{x}} x = \cos(\theta); \quad \partial_{\tilde{y}} x = -sin(\theta$$

Thus

$$\tilde{H}^2_2 \equiv \tilde{H}^y_y = H^x{}_x(\partial \tilde{y}/\partial_x)\partial x/\partial \tilde{y}) + H^y{}_x(\partial \tilde{y}/\partial y)\partial x/\partial \tilde{y}$$
$$= 1(-sin(\theta))(-sin(\theta) + (-1)(sin(x))(cos(\theta) \tag{16}$$

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2. Given that the metric is

$$ds^2 = dr^2 + r^2 d\phi^2_d z^2. \tag{17}$$

and the coordinates are $\{x^1, x^2, x^3\} \equiv \{r, \phi, z\}$, what is the inverse metric, the determinant of the metric, and the component of the anti-symmetric tensor ϵ^{123} Take $A_i = \{0, r^2, 0\}$ for $r < 1$ and is $\{0, 1, 0\}$ for $r > 1$, what are the components of B^i where $B^i = \epsilon^{ijk}\partial_j A_k$.

$$g_{rr} = g_{11} = 1; g_{\phi\phi} = g_{22} = r^2; \quad g_{zz} = g_{33} = 1 \tag{18}$$
$$g_{12} = g_{13} = g_{23} = g_{21} = g_{23} = g_{32} = g_{13} = g_{31} = 0 \tag{19}$$

This is a diagonal metric , and thus the determinant is the product of the three diagonal elements, namely r^2. The inverse metric is also diagonal and is the inverse of each of the three diagonal elemants or

$$g^{11} = g^{rr} = 1; \quad g^{22} = g^{\phi\phi} = \frac{1}{r^2}; \quad g^{33} = g^{zz} = 1$$

$$B^i = \epsilon^{ijk}\partial_j A_k \tag{20}$$
$$B^r = \frac{1}{\sqrt{g}} e^{rjk}\partial_j A_k = \frac{1}{r}(\partial_\phi A_z - \partial_z A_\phi) = 0 \tag{21}$$
$$B^\phi = \frac{1}{r}(\partial_z A_r - \partial_r A_z) = 0 \tag{22}$$
$$B^z = \frac{1}{r}(\partial_r A_\phi - \partial_\phi B_r) = \left\{ \begin{array}{ll} 2 & r < 1 \\ 0 & r > 1 \end{array} \right. \tag{23}$$

Ie, although A is non-sero everywhere (except r=0), B is non-zero only for r¡1. Ie, the vector potential is non-zero even where B is zero.
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3.Given the metric $ds^2 = dx^2 + dxdy + dy^2 + dz^2$ what are the components of the metric, the inverse metric and the determinant of the metric? $\{x^1, x^2, x^3\} \equiv \{x, y, z\}$

$$g_{xx} = g_{11} = 1; \quad g_{yy} = g_{22} = 1; \quad g_{33} = g_{zz} = 1 \tag{24}$$
$$g_{12} = g_{xy} = g_{21} = g_{yx} = \frac{1}{2} \tag{25}$$

Note that it is important that you remember that the off diagonal part given by $dxdy$ is shared between g_{xy} and g_{yx}. Einstein in 1913 got this wrong and wasted about a year of his life because he thought that his then equations did not satisfy the natural equations he derived and was driven to cook up another (wrong) theory.

The matrix is

$$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

. The determinant is $(1 \cdot 1 - (1/2)(1/2))1 = 3/4$. The inverse matrix is

$$\begin{pmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$g^{xx} = g^{yy} = 4/3; \quad g^{zz} = 1; \quad g^{xy} = g^{yx} = 2/3$$

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