Angular momentum functions

We are truing to solve the equation for the potential in spherical coordinates

$$\frac{1}{r^2}\partial_r r^2 \partial_r \Phi(r,\theta,\phi) + \frac{1}{r^2 sin(\theta}\partial_\theta sin(\theta)\partial_\theta \Phi + \frac{1}{r^2 sin(\theta)^2}\partial_\phi^2 = -\frac{1}{\epsilon_0}\rho(r,\theta,\phi) \tag{1}$$

We have already looked at the solution if we take the potential, and the charge density to be independent of angle– ie to be spherically symmetric. We got the two solutions for the empty space equations, $\frac{C}{r}$ and D where C, C are both constants. The first of these turned out not to be a solution for the zero charge situation, but rather represents a point charge at $r = 0$ It is also pathalogical solution in the sense that the total electric energy goes to infinity near $r = 0$. But this solution turned out to be extremely handy since one could use solutions centered at different points x', y', z' to form a Green;s function which can be used to turn the differential equation into an integral equation, which can be very handy.

However, what happens if the charge density in polar coordinates is not sperically symmetric, but rather has angular dependence $\rho(r, \theta, \phi)$? How can we solve this.

Let us first look at the Homogenous equation (ie, density equal to 0). The trick here is to make use of the linearity of the equations (ie, the sum of solutions is a solution) and a set of fiducial solutions, which are based on the symmetries of the equations.

We have the potential Φ obeying the above eqn 1. This is an interesting equation. because, for example, the equation does not, in the first place depend of ϕ at all. Furthermore the θ dependence is all sort of lumped together. Let us try to see if there are solutions (they are NOT the most general solutions) which are products of functions of the coordinates.

$$\Phi_{lm} = F(r)P(\theta)E(\phi). \tag{2}$$

put this anzatz into the equation with $\rho = 0$, and divide by Φ_{lm}. We get

$$\frac{\partial_r r^2 \partial_r F(r)}{F(r)} + \frac{\partial_\theta sin(\theta)\partial_\theta P(\theta)}{\sin(\theta)P(\theta)} + \frac{\partial_\phi^2 E(\phi)}{sin(\theta)^2 E(\theta)} = 0 \tag{3}$$

Now, the equation $\frac{\partial_\phi^2 E(\phi)}{E(\phi)} = -m^2$ is easily solved. It is just the equation for a simple Harmonic oscillator (where ϕ plays the role of time). Furthermore, the solution must be such that $E(\phi + 2\pi) = E(\phi)$ since going around by 2π brings us back to the same place. The solution is

$$E(\phi) = e^{im\phi} \tag{4}$$

where m must be a positive or negative integer so that $e^{im2\pi} = 1$. Let us now look at the θ equation. We have

$$\frac{1}{sin(\theta)}\partial_\theta sin(\theta)\partial_\theta P(\theta) - \frac{m^2}{sin(\theta)^2}P(\theta) = -l(l+1)P(\theta) \tag{5}$$

This is complicated equation, but we could make it a bit simpler if we defined $\xi = cos(\theta)$ so that $d\xi = sin(\theta)d\theta$. Then this equation becomes $P(\xi) = P(\cos(\theta))$ obeys

$$\partial_\xi(1-\xi^2)\partial_\xi P(\xi) - l(l+1)P(\xi) - \frac{m^2}{(1-\xi^2)}P(\xi) = 0 \tag{6}$$

The equation is somewhat pathological at $\xi = \pm 1$, so we have to choose the solutions such that the solutions are well behaved there. They have to have l and integer to be regular at $\theta = 0$ and π These equations have as regular solutions for $m > 0$

$$P_{l|m|}\xi = \frac{(-1)^l}{2^l l!}1 - \xi^{2^{|m|/2}}\partial_\xi^{l+|m|}(1-\xi^2)^l \tag{7}$$

(Note that this is usually written as

$$P_{lm} = \frac{(-1)^l}{2^l l!} 1 - \xi^{2|m|/2} \partial_\xi^{l+|m|} (x^2 - 1)^l \tag{8}$$

There are a number of other sign convetions A popular one is to add a $(-1)^m$ factor as well.

Note that since the function $(1 - x^2)^l$ has a maximum power of ξ to be ξ^{2l}, m cannot be greater than l. Letting $\xi = cos(\theta)$ and $\sqrt{1-\xi^2} = sin(\theta)$, we can find what these functions are in terms of θ These are the associate Legandre Polynomials.

Thus we have the angular functions

$$P(\theta)E(\phi) = P_{l|m|}(\theta)e^{im\phi} \tag{9}$$

and are left with the radial equations

$$\partial_r r^2 \partial_r F_l(r) - l(l+1)F_l(r) = 0 \tag{10}$$

or

$$F_l(r) = \frac{C_l}{r^{l+1}} + D_l r^l \tag{11}$$

where C_l and D_l are constants.

We note that near $r = \infty$, the D_l term goes to infinity, while near $r = 0$ the C_l term goes to infinity. Thus in free space, we find that all of the terms for $l > 0$ diverge either at $r = 0$ or at $r = \infty$, while for $l = 0$ one of the terms is that of a point source, while the other is a constant potential

The angular functions above can be chosen so that

$$Y_{lm}(\theta, \phi) = \alpha_{lm} P_{l|m|}(\theta) E_m(\phi) \tag{12}$$

so $Y^*_{lm} = Y_{l,-m}$ with α_{lm} chosen so that

$$\int_0^{2\pi} \int_0^{\pi} Y^*_{l'm'}(\theta, \phi) Y_{l'm'}(\theta, \phi) sin(\theta) d\theta d\phi = \delta_{l',l} \delta_{m,m'} \tag{13}$$

(This makes

$$\alpha_{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} \tag{14}$$

)

We can now find the solution to the equations by integrating the equation by assuming that

$$\Phi(r, \theta, \phi) = \sum_l \sum_{m=-l}^{l} \Phi_{lm}(r) Y_{lm}(\theta, \phi) \tag{15}$$

$$\rho(r, \theta, \phi) = \sum_l \sum_{m=-l}^{l} \rho_{lm}(r) Y_{lm}(\theta, \phi) \tag{16}$$

Then

$$(\frac{1}{r^2} \partial_r r^2 \partial_r - \frac{l(l+1)}{r^2}) \Phi(r, \theta, \phi) = 1\epsilon_0 \rho_{lm}(r) \tag{17}$$

By matching r^l with $r^{-(l+1})$ so they are continuous

$$G_{lm}(r, r') = -\frac{1}{\epsilon_0(2l+1)} \left(\frac{r^l}{r'^{l+1}} \Theta(r' - r) + \frac{r'^l}{r^{l+1}} \Theta(r - r') \right) \tag{18}$$

(The factor $2l+1$ comes from ensuring that $\partial_r^2\Phi_{lm}$ is equal to $\delta(r-r)$ We have that at $r=r'$ both terms are have equal amplitude, so the function is continuous, and for both $r>r'$ and $r<r'$ the radial equation on this function is zero, but the integral of

$$\int \frac{1}{r^2}\partial_r r^2 \partial_r - (\frac{l(l+1)}{r^2})G_{lm}(r,r')\frac{-f(r')}{\epsilon_0}r'^2 dr' = \frac{1}{\epsilon_0}f(r) \tag{19}$$

Ie,

$$G(\mathfrak{X},\mathfrak{X}') = \sum_l \sum_n G_{lm}(r,r')Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{20}$$

and

$$\Phi(r,\theta,\phi) = -\frac{1}{2\pi\epsilon_0}\int_r G_{lm}(r,r')Y_{lm}(\theta,\phi)(\int Y^*(\theta',\phi')\rho(r',\theta',\phi')sin(\theta')d\theta' d\phi')r'^2 dr' \tag{21}$$

Ie, one can convert a differntial equation into an integral equation and find the solution by quadratures. Note that near r=0, the solution will go as r^l, while near r=infinity, it will go as $1/r^{l+1}$. Ie, it will be regular at both limits.

Now

$$\Phi_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\Phi(r,\theta,\phi)Y^*_{lm}(\theta,\phi)\sin(\theta)d\theta d\phi \tag{22}$$

and similarly for $rho_{lm}(r)$.

These components are called the monopole moment for $l=0$, the dipole moment for $l=1$, the quadrapole moment for $l=2$, the octapole moment for $l=3$,... (Ie, greek number prefix of 2^l).

Note that each moment for greater r fall off faster and faster, so when far away only the lowest l with non-zero term is what will be seen.

As an example, let us take two point charges, located at $z=\pm a$ with opposite charges.

$$\rho(r,\theta,\phi) = q\delta(r-a)(\delta^2(\theta)-\delta^2(\theta-\pi) \tag{23}$$

where $\delta^2(\theta)$ is the two dimensional delta function at the singular point $\theta=0$, such that

$$\int_{\mathfrak{S}} f(\theta,\phi)\delta^2(\theta)d\mathfrak{S} = f(0,0)$$

For a smooth function over the surface, $f(0,\phi)$ would be independent of ϕ for $m\neq 0$ since the z axis is symmetric under ϕ translation. Then the integral of θ would only involve the $Y_{l0}(\theta)$

$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\left(\frac{r'^l}{r'^{l+1}}qY_{l0}(cos(\theta))(Y_{l0}(1)-Y_{l0}(-1))(\frac{a^l}{r^{l+1}}\Theta(r-a)+\frac{r^l}{a^{l+1}}\Theta(a-r)\right) \tag{24}$$

The $Y_{lm}(\theta,\phi)$ for even l are symmetric around $\theta=0$ and so the values at $\theta=0$ and π be the same and thus will cancel out. The odd values of l are antisymmetric. Each power of r^{-l} is called a multiple moment. of the distribution. If we shrink a, and expand q, such that qa is constant, then eventually as $a\to 0$, only the lowest order term (going as qa/r^2) will survive. This is the dipole term. The higher terms will go as $qa^l = qa(a^{l-1}$ which go to 0 as a goes to 0.

In the table are the Y_{lm} for the first three values of l. I have chosen the signs to be positive so that $Y^*_{lm} = Y_{l,-m}$ for all l and m. This sign convention does not really matter. Alternative sign conventions for the Y_{lm} can be based on the use of raising and lowering operators to define the Y_{lm} based on the the definiton of Y_{l0} defined in terms of the Legendre polynomials, or other requirements. They will in general not have the feature that $Y^*_{lm} = Y_{l,-m}$.

ℓ = 0

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

ℓ = 1

$$\begin{aligned}
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta &&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x-iy)}{r}\\
Y_1^{0}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta &&= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\frac{z}{r}\\
Y_1^{1}(\theta,\varphi) &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta &&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x+iy)}{r}
\end{aligned}$$

ℓ = 2

$$\begin{aligned}
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta &&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)^2}{r^2}\\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta &&= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)\cdot z}{r^2}\\
Y_2^{0}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) &&= \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot\frac{(3z^2-r^2)}{r^2}\\
Y_2^{1}(\theta,\varphi) &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta &&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)\cdot z}{r^2}\\
Y_2^{2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta &&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)^2}{r^2}
\end{aligned}$$

ℓ = 3

$$\begin{aligned}
Y_3^{-3}(\theta,\varphi) &= \frac{1}{8}\sqrt{\frac{35}{\pi}}\cdot e^{-3i\varphi}\cdot\sin^3\theta &&= \frac{1}{8}\sqrt{\frac{35}{\pi}}\cdot\frac{(x-iy)^3}{r^3}\\
Y_3^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta\cdot\cos\theta &&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot\frac{(x-iy)^2\cdot z}{r^3}\\
Y_3^{-1}(\theta,\varphi) &= \frac{1}{8}\sqrt{\frac{21}{\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot(5\cos^2\theta-1) &&= \frac{1}{8}\sqrt{\frac{21}{\pi}}\cdot\frac{(x-iy)\cdot(5z^2-r^2)}{r^3}\\
Y_3^{0}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{7}{\pi}}\cdot(5\cos^3\theta-3\cos\theta) &&= \frac{1}{4}\sqrt{\frac{7}{\pi}}\cdot\frac{(5z^3-3zr^2)}{r^3}\\
Y_3^{1}(\theta,\varphi) &= -\frac{1}{8}\sqrt{\frac{21}{\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot(5\cos^2\theta-1) &&= \frac{-1}{8}\sqrt{\frac{21}{\pi}}\cdot\frac{(x+iy)\cdot(5z^2-r^2)}{r^3}\\
Y_3^{2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta\cdot\cos\theta &&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\cdot\frac{(x+iy)^2\cdot z}{r^3}\\
Y_3^{3}(\theta,\varphi) &= \frac{1}{8}\sqrt{\frac{35}{\pi}}\cdot e^{3i\varphi}\cdot\sin^3\theta &&= \frac{-1}{8}\sqrt{\frac{35}{\pi}}\cdot\frac{(x+iy)^3}{r^3}
\end{aligned}$$

FIG. 1: Figure ylm. From https://en.wikipedia.org/wiki/Table_of_spherical_harmonics with signs altered to make $Y^*_{lm} = Y_{l,-m}$.