TEMPERATURE, PERIODICITY AND HORIZONS

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Contents

Abstract:

We explain and explore the connections among the following propositions: (1) thermal equilibrium is characterized by the KMS condition, $\langle A_{t-i\beta}B\rangle = \langle BA_t\rangle$; (2) finite-temperature Green functions are periodic in imaginary time; (3) black holes are hot; and to an accelerating observer, empty space is hot. The KMS condition of quantum statistical mechanics is derived, with special attention to quantum field systems satisfying relativistic commutation relations and linear field equations. We display the analytic structure of the two-point function and show in what sense the KMS condition for such systems is a statement of periodicity. Then the application of these ideas to horizons in general-relativistic settings is reviewed. Other matters discussed include: the identification of the analytically continued two-point function with the Green function of an elliptic ("Euclideanized") operator; the analogous relation between a nonrelativistic propagator and a parabolic operator; the construction of thermal two-point functions as image sums; the (in)significance of time ordering; simplifications of the KMS condition in the presence of discrete symmetries; the appearance of a "double" Fock space (artificially in general statistical mechanics, but naturally in space-times with horizons); and complications associated with the infrared behavior of the "particle" spectrum.

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1. Introduction

This article is an explication of the property of "periodicity in imaginary time" associated with states of finite temperature in quantum many-body theory and field theory, which in recent years has attracted much attention in connection with thermal effects related to black holes and "accelerated observers" [78]. Since we deal primarily with basic mathematical issues, our article is complementary to the review of Sciama et al. [78] (or the relevant chapters of the book of Birrell and Davies [7]), which is more concerned with the physical origin and implications of the effects.

In nonrelativistic quantum statistical mechanics (many-body theory) it has long been known [69, 70] that the Green functions, defined as finite-temperature expectation values of products of field operators, possess analytic continuations in the time variable, whose boundary values at $t=0$ and $t=\pm i\beta$ are related (essentially equal). "This relationship is crucial to all . . . Green's function analysis" [61]. This principle has penetrated various subcultures of physics in different forms. On the one hand, a generalization of it to arbitrary observables [52] has become a central tool of rigorous quantum statistical mechanics [59, 13], under the title of "KMS condition". On the other, it has recently taken root in special-relativistic quantum field theory (elementary particle theory) [14, 5, 32, 90, 2, 73] and in the general-relativistic quantum field theory previously mentioned [54, 49, 50, 35, 33, 34, 23, 80, 82]. The general-relativistic context is especially noteworthy because it provides a geometrical interpretation of the periodicity, which in other applications arises as a mathematical fact without much of an intuitive rationale.

Another formal mathematical property of thermal states of field or many-body systems is the appearance of a "doubled" Fock space as the natural Hilbert-space representation [1, 86]. Amazingly, this also has a clear geometrical origin in the general-relativistic settings [88, 60, 42, 65], which is closely related to the analyticity considerations.

In a special historical category are the papers of Höegh-Krohn et al. [57, 36] and Bisognano and Wichmann [8, 9], which provide – or should have provided – bridges between the literature of mathematical physics and that of general relativity. The relevance of the work of Bisognano and Wichmann, particularly, to that of Unruh [88] et al. was not appreciated until very recently [80]. (We should note that Ojima [73] draws on both the mathematical and the particle physics/statistical mechanics traditions.)

Most of the recent work on finite temperature in relativistic theories has taken as a starting point the well-developed literature of nonrelativistic many-body theory [e.g., 61]. The present authors became aware of a simple, direct derivation of the analytic and periodic properties of thermal Green functions in relativistic field theories, based on the local commutativity of the fields in analogy to the treatment of vacuum n-point functions in axiomatic field theory [e.g., 85]. In addition to simplifying the development of the theory, this observation allows its conclusions to be sharpened and strengthened. In particular, nowhere in the existing literature have we seen a clear account of what happens at the boundaries of the strips where the two-point function is analytic: In general there is a branch cut there (not just poles), and *the discontinuity across the cut is equal to the commutator function of the quantum field.*

We found other minor inadequacies in the physical literature; for example, the role of time ordering is usually quite unclear. On the other hand, the mathematical literature on the KMS condition is of little help in understanding the periodicity of the two-point function and its relation to the Green function of an operator on a "Euclidean" space (that is, one with positive definite metric). Indeed, the standard formulation of the KMS condition seems to state that a certain function of time is *not* periodic, but rather "periodic with a twist":

$$\langle A_{t-\mathrm{i}\beta} B \rangle_\beta = \langle B A_t \rangle_\beta \neq \langle A_t B \rangle_\beta \,.$$

The resolution of this apparent inconsistency is related to the presence of the branch cut mentioned above.

For these reasons we have prepared this expository article.

Section 2 is a detailed study of the two-point function of a scalar field with respect to states of zero temperature (vacuum) and of finite temperature. The field is "free" in the sense that it satisfies a linear field equation, but the formalism is broad enough to include external gravitational and magnetic potentials and various boundary conditions; in particular, the situations studied by the general relativists in connection with black holes and accelerating observers are covered. After a review of the vacuum case, the two-point function corresponding to the grand canonical ensemble at vanishing chemical potential and at temperature $1/\beta$ for a field theory in a finite region is constructed. Its analytic structure and iβ-periodicity are established, and also its connection with the Green function of a certain elliptic operator subject to periodic boundary conditions. By taking a limit ("the thermodynamic limit"), one obtains a function with these same properties which describes "finite temperature" for an infinite system – for which the usual grand-canonical-ensemble density matrix does not exist. We also discuss the "infrared problem" of fields whose normal-mode frequencies do not have a positive lower bound; whether the thermal two-point functions exist in such a case depends on the system, in a way linked to, but not determined by, the spatial dimensionality. Finally, we discuss the construction of the thermal function as an "image sum" of translates of the vacuum function.

In section 3 we turn to general quantum-theoretical systems and general observables, and derive the standard KMS condition. Then we show that commutativity of two observables (for some interval of time separations) enables one to extend the KMS function to an analytic, periodic function generalizing the two-point function of section 2. For two-point functions themselves, consequences of symmetry under charge conjugation and time reversal are investigated. We close with some comments on the significance of time ordering.

In section 4 we briefly review how finite-temperature states arise when fields near geometrical *horizons* are described in hyperbolic coordinates; this includes the famous cases of uniform acceleration and of black holes (the Schwarzschild–Kruskal metric). The Araki–Woods double-Fock-space construction (also known as "thermo-field dynamics" after Takahashi and Umezawa), which arises here naturally, is discussed. Since all horizon models have the infrared problem – even if the field is massive – we investigate the existence of the thermal states carefully.

Appendix A is devoted to the formulation of nonrelativistic quantum statistical mechanics within the framework established in the paper.

2. Relativistic "free" fields and elliptic Green functions

2.1. Definition of the physical system

We consider in detail a scalar field without self-interaction, in a geometrical setting sufficiently general to encompass many situations of interest.

Let M be an n-dimensional Riemannian manifold. For notational convenience, pretend that M can be covered by a single coordinate system, wherein the (positive definite) metric is given by

$$ds^2 = \gamma_{jk}(x)\,dx^j\,dx^k \tag{2.1}$$

(repeated Latin indices summed from 1 to n), and let

$$\gamma \equiv \det(\gamma_{jk})\,. \tag{2.2}$$

Define the scalar product

$$\langle \phi, \psi \rangle \equiv \int_{\mathrm{M}} \phi(x)^*\,\psi(x)\,\gamma(x)^{1/2}\,d^n x\,. \tag{2.3}$$

The corresponding Hilbert space of square-integrable functions is denoted by $L^2(\mathrm{M}; \gamma^{1/2})$ [or, when necessary for clarity, by $L^2(\mathrm{M}; \gamma^{1/2}\,d^n x)$].

Let K be the second-order differential operator (on scalar functions)

$$K = -\gamma^{jk}(x)\,[\nabla_j - iA_j(x)][\nabla_k - iA_k(x)] + V(x)\,, \tag{2.4}$$

where ∇ is the usual covariant derivative associated with the metric (2.1), γ^{jk} is the corresponding contravariant metric, and A and V are real-valued. We assume:

K, supplemented by (fixed but usually unspecified) boundary conditions if necessary, is a self-adjoint operator on the Hilbert space determined by (2.3); (2.5)

The spectrum of K is nonnegative; (2.6)

Zero is not an eigenvalue of K (but may be the lowest point of the continuous spectrum). (2.7)

(To avoid technical issues we also assume that the coefficient functions in (2.4) are smooth, although this is unnecessary for most of our considerations.) Our field will satisfy

$$-\partial^2\phi/\partial t^2 = K\phi \tag{2.8}$$

along with canonical commutation relations.

We have particularly in mind these cases:

(1) M is a region in $\mathbf{R}^n$, $\gamma_{jk}(x) = \delta_{jk}$, $A_j(x) = 0$, $V(x) = m^2$. Then $\phi(t, x)$ is the special-relativistic free field of mass m in $(n+1)$-dimensional space-time. If M is a rectangle

$$-L_j < x^j < L_j \qquad (0 < L_j \le \infty)\,, \tag{2.9}$$

then a pair of boundary conditions must be imposed for each L_j which is finite. If all L_j are finite and $m = 0$, then purely periodic or purely Neumann boundary conditions would violate (2.7), but Dirichlet conditions, for instance, are acceptable. If at least one L_j is infinite, the spectrum is continuous.

(2) $\mathcal{M} \equiv \mathbf{R} \times \mathrm{M}$ is a static $(n+1)$-dimensional space-time with metric $g_{\mu\nu}\,dx^\mu\,dx^\nu$ (Greek indices summed from 0 to n); $(g_{00})^{(1-n)/4}\phi$ satisfies the covariant Klein–Gordon equation on $\mathcal{M}$. Then (2.8)

holds with $A_j(x) = 0$,

$$\gamma_{jk}(x) = -g_{00}(x)^{-1}\, g_{jk}(x)\,,$$
$$V(x) = g_{00}(x)\, m^2 + \text{curvature terms}\,. \tag{2.10}$$

[The covariant Klein–Gordon equation,

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \tilde{\phi} + (m^2 + \xi R)\tilde{\phi} = 0\,,$$

is equivalent (in a static $\mathscr{M}$) to

$$-\partial^2\tilde{\phi}/\partial t^2 = \tilde{K}\tilde{\phi}\,,$$
$$\tilde{K}\tilde{\phi} \equiv g_{00}[g^{-1/2}\,\partial_j(g^{1/2} g^{jk}\partial_k \tilde{\phi}) + (m^2 + \xi R)\,\tilde{\phi}]\,. \tag{2.11}$$

$\tilde{K}$ is not of the form (2.4). However, the conformal transformation

$$\tilde{\phi} \equiv (g_{00})^{(1-n)/4}\phi \tag{2.12}$$

converts the equation to (2.8) with K defined by (2.4) and (2.10). The metric γ in (2.10) has come to be called the "optical metric", because its geodesics are the paths of photons (the spatial projections of the null geodesics of $\mathscr{M}$).]

External electrostatic potentials ($A_0 \neq 0$) and external gravitational fields which are stationary but not static ($g_{0j} \neq 0$) are not covered in this framework; they are best handled by passing to a first-order, two-component formalism for the scalar field [e.g., 62].

2.2. *The field operator and the vacuum state*

Here we review the standard quantization of the scalar field satisfying (2.8) [e.g., 11, 41, 43] and show how its two-point function is related by analytic continuation to the Green function of the elliptic operator

$$-\partial^2/\partial s^2 + K \tag{2.13}$$

on the manifold $\mathbf{R} \times \mathrm{M}$. We treat ϕ as a charged field; the neutral (Hermitian) case is similar but simpler.

We shall assume temporarily that K has purely discrete spectrum; in this subsection (in contrast to the next) this is little more than a notational convenience. Accordingly, let $\psi_\nu(x)$ ($\nu = 1, 2, \ldots$) be the normalized eigenfunctions:

$$K\psi_\nu = \omega_\nu^2 \psi_\nu\,, \qquad \langle \psi_\nu, \psi_\nu \rangle = 1\,, \tag{2.14}$$

$\inf(\omega_\nu) > 0$. Then the quantized field is

$$\phi(t, x) = \sum_{\nu=1}^{\infty} \psi_\nu(x)\,(2\omega_\nu)^{-1/2}[\exp(-\mathrm{i}\omega_\nu t)\, a_\nu + \exp(\mathrm{i}\omega_\nu t)\, b_\nu^\dagger]\,, \tag{2.15}$$

where a_ν annihilates a quantum, $b_\nu^\dagger$ creates an antiquantum, and $[a_\nu, a_\mu^\dagger] = \delta_{\mu\nu}$, etc. [See also remark containing (3.17).] If

$$t_2 - t_1 \equiv t\,, \tag{2.16}$$

one has

$$[\phi(t_2, x), \phi(t_1, y)] = 0\,,$$

$$[\phi(t_2, x), \phi^\dagger(t_1, y)] = [\phi(t, x), \phi^\dagger(0, y)] = \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1} (\exp(-\mathrm{i}\omega_\nu t) - \exp(\mathrm{i}\omega_\nu t))\,, \tag{2.17}$$

$$[\phi(t_2, x), \phi^\dagger(t_1, y)] = 0 \qquad \text{if } (t_2, x) \text{ and } (t_1, y) \text{ have spacelike separation}\,. \tag{2.18}$$

(It is not obvious that (2.18) follows from (2.17); rather, (2.18) is a consequence of the canonical commutation relations and the finite propagation speed of the hyperbolic equation (2.8) [e.g., 29].)

The field operator (2.15) is an operator-valued distribution acting on a Fock space generated by a vacuum vector, $|0\rangle$, annihilated by all the a_ν and b_ν. Since the field equation is linear, the vacuum state is completely characterized by its (Wightman) two-point function,

$$\begin{aligned} G_+^\infty(t, x, y) &\equiv \langle 0|\phi(t_2, x)\, \phi^\dagger(t_1, y)|0\rangle \\ &= \langle 0|\phi(t, x)\, \phi^\dagger(0, y)|0\rangle = \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1} \exp(-\mathrm{i}\omega_\nu t)\,. \end{aligned} \tag{2.19}$$

[We have $\langle 0|\phi(t_2, x)\, \phi(t_1, y)|0\rangle = 0$ and the same for $\phi^\dagger$. Also, $\langle 0|\phi^\dagger(t, x)\, \phi(0, y)|0\rangle$ equals $\langle 0|\phi(-t, x)\, \phi^\dagger(0, y)|0\rangle^*$; this expresses invariance under combined charge conjugation and time reversal (see section 3.3).] It is convenient to define also

$$G_-^\infty(t, x, y) \equiv \langle 0|\phi^\dagger(t_1, y)\, \phi(t_2, x)|0\rangle = \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1} \exp(+\mathrm{i}\omega_\nu t)\,. \tag{2.20}$$

Note that the commutator (2.17) equals $G_+^\infty - G_-^\infty$. The superscript "∞" indicates that the vacuum state has zero temperature, hence reciprocal temperature $\beta = 1/T = \infty$.

Now we introduce a complex variable

$$z = t + \mathrm{i}s\,, \tag{2.21}$$

t and s real, and investigate the analytic continuations of $G_\pm^\infty$ off the real z axis. By virtue of the extra exponential decay of the summand in the respective region, (2.19) with t replaced by z defines a holomorphic function in the lower half plane, and (2.20) defines a holomorphic function in the upper half plane. The distributions $G_\pm^\infty$ are the boundary values of these functions as the real axis is approached from their respective directions.

Remark: The asserted exponential decay follows from the fact that the spectral function

$$E_\lambda(x, y) = \sum_{\omega_\nu < \lambda} \psi_\nu(x)\, \psi_\nu(x)^* \tag{2.22}$$

is polynomially bounded as $\lambda \to \infty$. Indeed, under our assumptions on M and K it is known [44, 46, 58] that $E_\lambda(x, y)$ is of order λ^n (not only for an operator with discrete spectrum but also in the case of continuous spectrum, which we consider later). For many solvable models (such as when M is a rectangle and K is the Laplacian) this fact is well known and easily verified. It may be helpful to sketch a few steps of the proof in the general case: First, one shows that the spectral projection E_λ has a continuous kernel, $E_\lambda(x, y)$. Invoking the Schwarz inequality for the sesquilinear form $\langle f, E_\lambda g\rangle$, one then infers

$$|E_\lambda(x, y)|^2 \leq E_\lambda(x, x)\, E_\lambda(y, y)\,.$$

Thus one needs only consider $E_\lambda(x, x)$, whose Laplace transform

$$\int_0^\infty \exp(-\lambda^2 t)\, \mathrm{d}E_\lambda(x, x)$$

is known to go like $c_1 t^{-n/2}$ for $t \downarrow 0$. (This is the first term of the well-known heat-kernel expansion.) The Karamata Tauberian theorem [91, section 5.4] then yields $E_\lambda(x, x) \sim c_2 \lambda^n$ for $\lambda \to \infty$.

If $x \neq y$, then the separation of (t_2, x) and (t_1, y) will be spacelike for sufficiently small t in (2.16) (namely, t less than $d(x, y)$, the distance between x and y in the metric γ_{jk}). Thus (2.18) implies that

$$G_+^\infty(z, x, y) = G_-^\infty(z, x, y)$$

for z on a certain interval $(-d < t < d)$ of the real axis. By the one-dimensional edge-of-the-wedge theorem [85, section 2.5], therefore, each of these functions is an analytic continuation of the other. That is, for fixed, unequal x and y there is a single holomorphic function, $\mathcal{G}^\infty(z, x, y)$, defined on a connected region of the complex plane (see fig. 1), such that

$$\mathcal{G}^\infty(z, x, y) = \begin{cases} G_+^\infty(z, x, y) & \text{if} \quad \operatorname{Im} z < 0\,, \\ G_-^\infty(z, x, y) & \text{if} \quad \operatorname{Im} z > 0\,, \end{cases} \tag{2.23}$$

and both equalities hold if $\operatorname{Im} z = 0$ and $|\operatorname{Re} z| < d$. In general there will be branch cuts along the real axis from $z = \pm d$ to $z = \pm\infty$.

Now consider the values of $\mathcal{G}^\infty(z, x, y)$ on the imaginary axis. If $z = \mathrm{i}s$, then from (2.23) and (2.19–20) we have

$$G^\infty(s, x, y) \equiv \mathcal{G}^\infty(\mathrm{i}s, x, y) = \sum_{\nu=1}^\infty \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1} \exp(-\omega_\nu |s|)\,. \tag{2.24}$$

Thus, we can also write

$$G^\infty(s, x, y) = (2\pi)^{-1} \sum_{\nu=1}^\infty \int_{-\infty}^\infty \mathrm{d}k_0\, \psi_\nu(x) \exp(\mathrm{i}k_0 s_2)\, \psi_\nu(y)^* \exp(-\mathrm{i}k_0 s_1)\, (k_0^2 + \omega_\nu^2)^{-1} \tag{2.25}$$

where s_1 and s_2 are real numbers such that $s_2 - s_1 = s$. G^∞ is often called the "(two-point) Schwinger

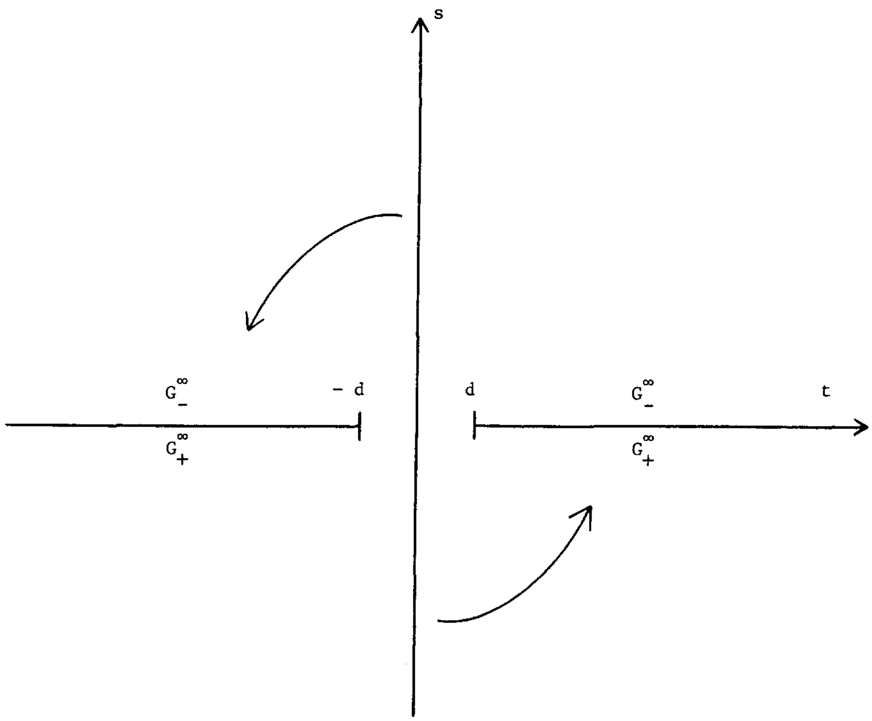

Fig. 1.

function" of the scalar field ϕ. It is clear from (2.25) [recall the normalization (2.14), (2.3)] that G^∞ is the integral kernel of the inverse of (2.13) (viewed as an operator on $L^2(\mathbf{R}\times M; \gamma^{1/2}\, ds\, d^n x)$). Equivalently, G^∞ may be characterized as the Green function of (2.13), acting on $\mathbf{R}\times M$. That is, G^∞ is the unique solution to

$$(-\partial^2/\partial s_2^2 + K_{(x)})G = \delta(s_2 - s_1)\,\delta(x-y)\,\gamma(y)^{-1/2} \tag{2.26}$$

which decays as $|s|\to\infty$.

Consequently, if one starts from the Green function $G^\infty(s, x, y)$ of the elliptic or "Euclideanized" problem, then by analytic continuation in s one can reach either of the space-time Wightman functions $G_+^\infty(t, x, y)$ and $G_-^\infty(t, x, y)$—which are solutions of the related homogeneous, hyperbolic equation

$$(+\partial^2/\partial t_2^2 + K_{(x)})G_\pm^\infty = 0\,. \tag{2.27}$$

From fig. 1, reoriented, one sees that G_+ is obtained by approaching the imaginary s axis from the left, G_- by approaching from the right.

Note that up to now we have said *nothing* about time ordering. The time-ordered (Feynman) two-point function is

$$G_F^\infty(t, x, y) \equiv \langle 0|\mathcal{T}[\phi(t_2, x)\,\phi^\dagger(t_1, y)]|0\rangle = \begin{cases} G_+^\infty(t, x, y) & \text{if} \quad t_2 > t_1\,, \\ G_-^\infty(t, x, y) & \text{if} \quad t_2 < t_1\,; \end{cases} \tag{2.28}$$

$\mathcal{T}[\ldots]$ denotes the product of field operators ordered with time arguments increasing to the left. G_{F}^{∞} can be obtained from G^{∞} by a *rigid* rotation of the domain from the s axis to the t axis in the direction indicated by the curved arrows in fig. 1:

$$G_{\mathrm{F}}^{\infty}(t, x, y) = G^{\infty}(-\mathrm{i}t, x, y) = \sum_{\nu=1}^{\infty} \psi_{\nu}(x)\, \psi_{\nu}(y)^{*}\, (2\omega_{\nu})^{-1} \exp(-\mathrm{i}\omega_{\nu}|t|)\,. \tag{2.29}$$

It is only in this sense that G_{F}^{∞} has any closer relationship than the other Minkowski two-point functions to the Euclidean Green function, G^{∞}.

If K possesses some continuous spectrum, then the sums over ν in the preceding are replaced by Stieltjes integrals with respect to a spectral function E_{λ} generalizing (2.22) [45]. The only essential complication arises when zero is a limit point of the spectrum, as is expressly allowed by (2.7).

We shall now illustrate this and the preceding developments by considering the free scalar field in flat space-times. First look at Minkowski space, $\mathbf{R}^{d} \equiv \mathbf{R}^{n+1}$, and $K \equiv -\nabla^{2} + m^{2}$ acting in $\mathbf{R}^{n}$. Then the Schwinger function (2.24) becomes

$$\mathcal{G}^{\infty}(\mathrm{i}s, x, y) = S_{n+1}([s^{2} + (x-y)^{2}]^{1/2}; m^{2})\,, \tag{2.30}$$

$$S_{d}(r; m^{2}) \equiv (2\pi)^{-d} \int \mathrm{d}^{d}k \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\, (k^{2} + m^{2})^{-1}\,. \tag{2.31}$$

[In (2.30), $x \in \mathbf{R}^{n}$; in (2.31), $\boldsymbol{x} \in \mathbf{R}^{d}$ and $r \equiv |\boldsymbol{x}|$.] For fixed $x \neq y$, one can read off the behavior of $\mathcal{G}^{\infty}(z, x, y)$ on the cuts from this. Recall that for $m > 0$ and d even, S_{d} is a combination of powers and modified Bessel functions, and for d odd it is a combination of exponentials and powers; in particular, $S_{3}(r; m^{2})$ is a multiple of the Yukawa potential, $r^{-1}\exp(-mr)$. Thus the only singularities on the Riemann surface of $\mathcal{G}^{\infty}$ are the points $z = \pm|x-y|$, and these are either infinite- or first-order branch points. Also,

$$S_{d}(r; 0) \propto r^{-d+2} \qquad \text{if } d > 2\,,$$

so that one gets square-root branch points for $m = 0$ when n is even. However, when $n = 3, 5, 7, \ldots$, there is only one sheet, since the two singularities reduce to poles. Note that this is equivalent to the commutator's vanishing in the timelike region (Huygens's principle).

For $m = 0$ and $d = 2$ (or $d = 1$) one has a nonintegrable singularity at the origin in (2.31). The implications of this for two-dimensional quantum field theory have been discussed in [77, 92 (section 4), 31]. The divergence reflects the fact that "$1/k^{2}$" defines a distribution only on test functions whose support (in k-space) does not include the origin. There are infinitely many ways to extend it to arbitrary test functions. These extensions correspond precisely to the various choices of a fundamental solution for the Laplacian in dimension 2. Recall that for $d > 2$ the fundamental solution is uniquely determined by requiring that it decay at ∞, but for $d \leq 2$ no such solutions exist. One may take, e.g.,

$$S_{2}(r; 0) \equiv -\frac{1}{2\pi} \ln\!\left(\frac{r}{R}\right) \tag{2.32}$$

for an arbitrary length R, but no choice of R produces decay. More to the point for field theory, no choice of R gives S_2 the positivity property necessary for reconstruction of field operators, in a Hilbert space with positive definite metric, defined on all smooth test functions.

There is a spectrum of attitudes one may take toward this situation. *Conservative:* An infrared divergence in the formal expression for the two-point function indicates that no quantum field theory exists; the model is inconsistent. *Radical:* The state space of such a system has an indefinite metric, and one must learn how to make physical sense of such a situation. *Liberal:* A quantum field theory exists, in a genuine Hilbert space, but the field operators are not defined for all test functions. We adopt the liberal point of view. Within that framework there are two policies one could adopt toward the two-point function: (1) No two-point function exists for such a theory. (2) As two-point function one may choose any "regularization" (i.e., an analogue of (2.32) for the model in question), remembering that its value on the illicit test functions is irrelevant to physics. The choice between (1) and (2) is perhaps a matter of taste and semantics; we shall adopt the former, because it relieves us of any obligation to discuss technical issues connected with regularizations, and because "the two-point function does not exist" is a convenient way to refer to the infrared pathology in passing.

The examples just discussed show that the presence of zero in the continuous spectrum is a necessary but not sufficient condition for the infrared pathology. To explore this subject further, let us now consider the rectangles (2.9) with $m = 0$. Suppose that $L_j = \infty$ for $j = 1, \ldots, J$ and $L_j < \infty$ for $j = L+1, \ldots, n$. Then (2.25) becomes (with $x^0 \equiv s_2$, $y^0 \equiv s_1$)

$$G^{\infty}(s, x, y) \equiv (2\pi)^{-(J+1)} \int_{-\infty}^{\infty} \mathrm{d}k_0 \cdots \int_{-\infty}^{\infty} \mathrm{d}k_J \sum_{k_{J+1}} \cdots \sum_{k_n} \prod_{j=J+1}^{n} (2L_j)^{-1} \times \exp\Big(\mathrm{i} \sum_{j=0}^{n} k_j x^j\Big) \exp\Big(-\mathrm{i} \sum_{j=0}^{n} k_j y^j\Big) \Big(\sum_{j=0}^{n} k_j^2\Big)^{-1}, \tag{2.33a}$$

where the precise spectra of the discrete variables $k_{J+1}, \ldots, k_n$ depend on boundary conditions. Each of these will take on the value $k_j = 0$ if the corresponding boundary conditions are of the periodic or Neumann type. Therefore, the integral (2.33a) will be infrared-divergent in those cases (even after the implicit smearing with smooth test functions, which handles any problem at the ultraviolet end) if and only if $J = 1$. (The case $J = 0$ has been ruled out by (2.7).) The same observation applies to

$$G^{\infty}(s, x, y) \equiv (2\pi)^{-J} \int_{-\infty}^{\infty} \mathrm{d}k_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}k_J \sum_{k_{J+1}} \cdots \sum_{k_n} \prod_{j=J+1}^{n} (2L_j)^{-1} \times \exp\Big[\mathrm{i} \sum_{j=1}^{n} k_j (x^j - y^j)\Big] \frac{1}{2} \Big(\sum_{j=1}^{n} k_j^2\Big)^{-1/2} \exp\Big[-\Big(\sum_{j=1}^{n} k_j^2\Big)^{-1/2} |s|\Big], \tag{2.33b}$$

corresponding to (2.24), and to the corresponding formulas (2.19–20) for $G_{\pm}^{\infty}$. So the two-point function fails to exist, in either the physical or the Euclidean domain, for the massless scalar field in two-dimensional Minkowski space ($J = n = 1$), as already mentioned, or in an infinite cylinder of square cross section in four-dimensional Minkowski space ($J = 1$, $n = 3$) with boundary conditions of the indicated types, etc.

In general, infrared convergence is governed by the behavior as $\lambda \downarrow 0$ of the spectral function, $E_\lambda(x, y)$. In concrete cases which can be solved by separation of variables, this can be investigated rather directly (see section 4.3).

Remark: The crux of the infrared-convergence issue is the following [cf. 40 (p. 247), 62, 43]: The two-point function will exist, as a distribution in (x, y) with t fixed, if and only if the field operator is defined as a distribution in x on test functions in the corresponding space, say $C_0^\infty(\mathrm{M})$. From the field expansion (2.15) generalized to continuous spectrum, one sees that $f(x)$ is an acceptable test function for ϕ if and only if the spectral transform

$$\omega^{-1/2}\,\tilde{f}(\omega) \equiv \omega^{-1/2} \int_{\mathrm{M}} \psi_\omega(x)^*\, f(x)\, \gamma(x)^{1/2}\, \mathrm{d}^n x \tag{2.34}$$

is an acceptable test function for the canonical annihilation and creation operators, $a(\tilde{g})$ and $b(\tilde{g})$. But in the Fock representation those operators are defined precisely for functions $\tilde{g}(\omega)$ which are square-integrable. Thus one needs f to be in the domain of $K^{-1/4}$ as an operator in the Hilbert space $\mathrm{L}^2(\mathrm{M}; \gamma^{1/2})$. *Infrared divergence occurs when not all functions in* $C_0^\infty(\mathrm{M})$ *belong to* $\mathrm{dom}(K^{-1/4})$. Elementary power counting shows that this criterion is consistent with our conclusions above for the massless free field in rectangular regions. Comparing (2.34) with (2.24) or (2.19–20), one sees how this connection will extend to the general case; we return to this in a remark in section 2.4.

2.3. Thermal states "in a box"

The thermal equilibrium state of temperature T corresponding to a time-independent Hamiltonian H is customarily defined by the Gibbs formula

$$\langle A \rangle_\beta = \mathrm{Tr}(\mathrm{e}^{-\beta H} A)/\mathrm{Tr}(\mathrm{e}^{-\beta H}) \tag{2.35}$$

for the expectation of an observable A, where

$$\beta = 1/T\,. \tag{2.36}$$

(We adopt units where Boltzmann's and Planck's constants are equal to unity.) However, for the numerator and denominator in (2.35) to be separately defined, it is *essential* that $\mathrm{e}^{-\beta H}$ be an operator of trace class. That is, H must have purely point spectrum $\{E_\sigma\}$, and the convergence condition

$$Z \equiv \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \sum_{\sigma=1}^{\infty} \mathrm{e}^{-\beta E_\sigma} < \infty \tag{2.37}$$

must be satisfied. Thermal states for more general systems may be definable as limits of the states of form (2.35) of approximating systems with finite Z.

For the relativistic system with equation of motion (2.8), the Hamiltonian H is the second-quantization of $K^{1/2}$. It is well known [e.g., 44, 24, 58] that if M is *compact* (with or without boundary), then the differential operator K (2.4) has a discrete spectrum and the number of eigenvalues less than or equal to λ^2 asymptotically approaches a constant times λ^n; thus the trace condition (2.37) is satisfied, as shown below.

Remark: If M is not compact, then (2.37) may diverge even if the spectrum is discrete. If $K = -\nabla^2 + V(x)$ on $\mathbf{R}^n$ with $V(x) \to +\infty$ as $|x| \to \infty$, then [75, 24] the number of eigenvalues less than λ^2 is asymptotic to

$$\text{const.} \int\limits_{V(x)<\lambda^2} [\lambda^2 - V(x)]^{n/2}\, d^n x\,. \tag{2.38}$$

If $V(x)$ approaches ∞ sufficiently slowly, then (2.38) grows arbitrarily rapidly – hence possibly faster than exponentially, so that (2.37) diverges. (On the other hand, the local spectral function (2.22) satisfies a polynomial bound, under very general conditions on V, so (2.43) and (2.50) will converge for such systems. This qualitative difference between the local and global spectral asymptotics is explained by the observation that normalized eigenfunctions of large eigenvalue ω_ν^2 will typically be small everywhere except near the "turning manifold" $V(x) \approx \omega_\nu^2$, which recedes to infinity as ω_ν increases.)

Accordingly, in the general case one starts by "cutting off" M by inserting a boundary, $\partial\Lambda$, whose interior Λ is compact. At the end (see section 2.4) one investigates the limit of (2.35) for a nested sequence of such "boxes" Λ whose union is all of M. Care must be taken to ensure that the cut-off systems satisfy (2.5–7). This can be done for many models by imposing the Dirichlet boundary condition on $\partial\Lambda - \partial\mathrm{M}$.

We now define thermal two-point functions by inserting the product of two fields into the formula (2.35) in the role of A:

$$\begin{aligned} G_+^\beta(t, x, y) &\equiv Z^{-1}\,\mathrm{Tr}[e^{-\beta H}\, \phi(t_2, x)\, \phi^\dagger(t_1, y)] \\ &\equiv Z^{-1}\,\mathrm{Tr}[e^{-\beta H}\, \phi(t, x)\, \phi^\dagger(0, y)] \end{aligned} \tag{2.39}$$

(where $t \equiv t_2 - t_1$ as always), and similarly

$$G_-^\beta(t, x, y) \equiv Z^{-1}\,\mathrm{Tr}[e^{-\beta H}\, \phi^\dagger(0, y)\, \phi(t, x)]\,. \tag{2.40}$$

For our field-in-a-box these quantities exist in the Fock representation and can be evaluated from (2.15): From (2.37),

$$Z = \sum_{\substack{n_1, n_2, \ldots = 0 \\ \bar n_1, \bar n_2, \ldots = 0}}^{\infty} \exp\left[-\beta \sum_{\nu=1}^{\infty} (n_\nu + \bar n_\nu)\, \omega_\nu\right]$$

(where n_ν and $\bar n_\nu$ are the numbers of quanta and antiquanta in mode ν)

$$\begin{aligned} &= \prod_{\nu=1}^{\infty} \left[\sum_{n_\nu=0}^{\infty} \exp(-\beta n_\nu \omega_\nu)\right]^2 = \prod_{\nu=1}^{\infty} (1 - \exp(-\beta\omega_\nu))^{-2} \\ &= \exp\left[-2 \sum_{\nu=1}^{\infty} \ln(1 - \exp(-\beta\omega_\nu))\right] \equiv [\det(1 - \exp(-\beta K^{1/2})]^{-2}\,. \end{aligned}$$

The series is convergent, since K was constructed so that $\omega_\nu > 0$ and $\Sigma_{\nu=1}^{\infty} \exp(-\beta\omega_\nu) < \infty$. Thus

$$Z^{-1} = \prod_{\nu=1}^{\infty} (1 - \exp(-\beta\omega_\nu))^2 \tag{2.41}$$

for a charged field. For a neutral field the exponent 2 is absent. Also,

$$\langle a_\nu^\dagger a_\nu \rangle_\beta \equiv Z^{-1}\,\mathrm{Tr}(\mathrm{e}^{-\beta H} a_\nu^\dagger a_\nu) = Z^{-1} \sum_{\text{all } n,\, \bar{n}} \exp\left[-\beta \sum_{\mu=1}^{\infty} (n_\mu + \bar{n}_\mu)\omega_\mu\right] n_\nu$$

$$= (1-\exp(-\beta\omega_\nu)) \sum_{n_\nu=0}^{\infty} n_\nu \exp(-\beta n_\nu \omega_\nu) = -(1-\exp(-\beta\omega_\nu))\, \omega_\nu^{-1} \frac{\partial}{\partial\beta} \sum_{n=0}^{\infty} \exp(-\beta n \omega_\nu)\,,$$

whence

$$\langle a_\nu^\dagger a_\nu \rangle_\beta = \exp(-\beta\omega_\nu)\,(1-\exp(-\beta\omega_\nu))^{-1}\,, \tag{2.42a}$$

$$\langle a_\nu a_\nu^\dagger \rangle_\beta = 1 + \langle a_\nu^\dagger a_\nu \rangle_\beta = (1-\exp(-\beta\omega_\nu))^{-1}\,. \tag{2.42b}$$

Identical equations hold for b_ν. The other combinations, $\langle a_\nu^\dagger a_{\nu'} \rangle_\beta$, $\langle a_\nu a_\nu \rangle_\beta$, $\langle a_\nu b_\nu^\dagger \rangle_\beta$, etc., are zero. Thus, finally,

$$G_+^\beta(t, x, y) = \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1}\, [\exp(-\mathrm{i}\omega_\nu t)\, \langle a_\nu a_\nu^\dagger \rangle_\beta + \exp(\mathrm{i}\omega_\nu t)\, \langle b_\nu^\dagger b_\nu \rangle_\beta]\,,$$

with a similar formula for G_-; hence

$$G_\pm^\beta(t, x, y) = \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1}\, (1-\exp(-\beta\omega_\nu))^{-1}$$
$$\times [\exp(\mp\mathrm{i}\omega_\nu t) + \exp(\pm\mathrm{i}\omega_\nu t)\exp(-\omega_\nu\beta)]\,. \tag{2.43}$$

As a check, note that

$$G_+^\beta(t, x, y) - G_-^\beta(t, x, y) = [\phi(t, x), \phi^\dagger(0, y)] \tag{2.44}$$

is consistent with (2.17).

Now let us set $z \equiv t + \mathrm{i}s$. Since the factor

$$q_+(z) \equiv \exp(-\mathrm{i}\omega_\nu z) + \exp(\mathrm{i}\omega_\nu (z + \mathrm{i}\beta)) \tag{2.45}$$

decreases exponentially with ω_ν if (and only if) $-\beta < s < 0$, $(2.43)_+$ with t replaced by z defines a holomorphic function

$$G_+^\beta(z, x, y) \qquad \text{for } -\beta < \operatorname{Im} z < 0\,. \tag{2.46a}$$

Similarly, $(2.43)_-$ yields a holomorphic extension

$$G_-^\beta(z, x, y) \qquad \text{for } 0 < \operatorname{Im} z < \beta \tag{2.46b}$$

corresponding to the function

$$q_-(z) \equiv \exp(\mathrm{i}\omega_\nu z) + \exp(-\mathrm{i}\omega_\nu (z - \mathrm{i}\beta))\,. \tag{2.47}$$

When $x \neq y$, for exactly the same reason as in the zero-temperature case, G_+^β and G_-^β are analytic continuations of each other through a window in the real axis as in fig. 1. Furthermore, from (2.45) and (2.47) one has

$$q_+(z - i\beta) = q_-(z)\,, \tag{2.48}$$

which means that the function in the upper strip, (2.46b), is an exact copy of that in the lower strip, (2.46a). Therefore, the process of analytic continuation can be repreated indefinitely in both directions. The result is a function $\mathscr{G}^\beta(z, x, y)$ which is holomorphic in the z plane except for horizontal branch cuts from $\mathrm{Re}\, z = \pm d(x, y)$ to $\mathrm{Re}\, z = \pm\infty$ at $\mathrm{Im}\, z = N\beta$ $(N = 0, \pm 1, \ldots)$; see fig. 2. The function satisfies the periodicity condition

$$\mathscr{G}(z + iN\beta, x, y) = \mathscr{G}(z, x, y) \text{ for all integers } N\,. \tag{2.49}$$

[If z is on one of the cuts, one must distinguish the limiting values of $\mathscr{G}$ from above and below the cut. Equation (2.49) is still valid for these boundary values, but to interpret it correctly one must remember the necessity of "jumping over" a cut. See also section 3.3.]

The analogue of (2.24) is

$$\begin{aligned} G^\beta(s, x, y) &\equiv \mathscr{G}^\beta(is, x, y) \\ &= \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^*\, (2\omega_\nu)^{-1}\, (1 - \exp(-\beta\omega_\nu))^{-1}\, [\exp(-\omega_\nu s) + \exp(+\omega_\nu(s - \beta))] \\ &\quad \text{if } 0 < s < \beta\,. \end{aligned} \tag{2.50a}$$

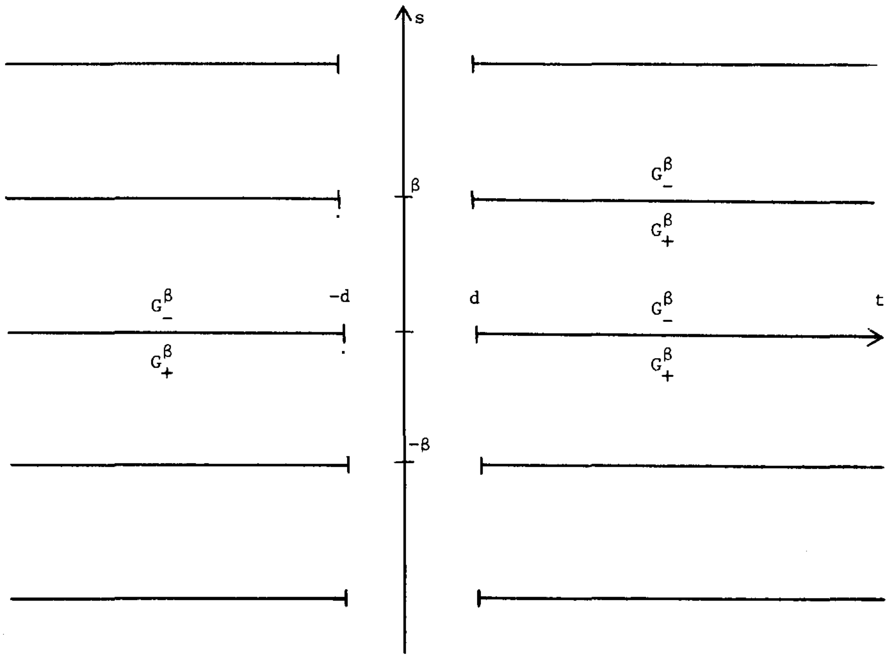

Fig. 2.

[Replacing s by $|s|$ in the final bracket, one gets an expression valid for $-\beta < s < \beta$; see (2.43). Thus G^β is even in s, just as in the $T=0$ case. Note that its periodicity (2.49) then implies that it is also symmetric with respect to reflection at the lines $\mathrm{Im}\, z = (N+\frac{1}{2})\beta$.] We claim that G^β is the solution of (2.26) which satisfies $G^\beta(s+\beta, x, y) = G^\beta(s, x, y)$; that is, G^β is the Green function for the elliptic operator (2.13) acting on a "cylinder" $\mathrm{S}^1 \times \Lambda$ of circumference β! The eigenfunction expansion of that Green function, parallel to (2.25), is (for $0 < s_2 - s_1 < \beta$)

$$G^\beta(s, x, y) = \beta^{-1} \sum_{\nu=1}^{\infty} \sum_{n=-\infty}^{\infty} \psi_\nu(x) \exp(2\pi \mathrm{i} n s_2/\beta)\, \psi_\nu(y)^* \exp(-2\pi \mathrm{i} n s_1/\beta) \left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega_\nu^2\right]^{-1} . \tag{2.50b}$$

So the proof of the claim reduces to the verification of

$$\sum_{n=-\infty}^{\infty} \left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]^{-1} \exp(2\pi \mathrm{i} n s/\beta) = \beta(2\omega)^{-1}(1-\mathrm{e}^{-\beta\omega})^{-1}[\mathrm{e}^{-\omega s} + \mathrm{e}^{\omega(s-\beta)}] \qquad \text{for } 0 < s < \beta . \tag{2.51}$$

The coefficients of this Fourier series can be verified by an elementary integration.

As in (2.28–29), a time-ordered thermal two-point function can be defined, which satisfies

$$G_{\mathrm{F}}^\beta(t, x, y) = G^\beta(-\mathrm{i}t, x, y) ; \tag{2.52}$$

that is, the real axis is approached from above if $t < 0$ and from below if $t > 0$. Note that it may be written

$$G_{\mathrm{F}}^\beta(t, x, y) = G_-^\beta(t, x, y) + \theta(t)\,[\phi(t, x), \phi^\dagger(0, y)] . \tag{2.53}$$

Hence, it satisfies the inhomogeneous wave equation

$$[\partial^2/\partial t^2 + K_{(x)}]G = -\mathrm{i}\delta(t)\,\delta(x-y)\,\gamma(y)^{-1/2} . \tag{2.54}$$

2.4. Thermal states in the whole space

We now consider taking the box, $\partial\Lambda$, away to infinity. It is natural to expect – although not obvious – that G^β for $\mathrm{S}^1 \times \Lambda$ will converge (as a distribution) to the corresponding Green function on $\mathrm{S}^1 \times \mathrm{M}$. [This would show, in particular, that the limit is independent of the shapes of the boxes in the particular sequence of boxes chosen. Similarly, one expects the same limiting result if the Dirichlet condition on $\partial\Lambda$ is replaced by any condition maintaining self-adjointness and positivity.] For the Green function on $\mathrm{S}^1 \times \mathrm{M}$ formulas (2.50) still apply, if the sum is replaced by the appropriate Stieltjes integral; the corresponding generalizations of formulas (2.43) remain meaningful; and the argument leading from (2.43) and (2.44) to (2.50) can be reversed. Thus G^β on $\mathrm{S}^1 \times \mathrm{M}$ has an analytic continuation to two-point functions $G_\pm^\beta$ (and G_{F}^β) on the space-time $\mathbf{R} \times \mathrm{M}$, which can be used to define the thermal equilibrium state for temperature $1/\beta$, and the master analytic function $\mathscr{G}$ for that system will exhibit the periodicity (2.49).

This state, as a functional on the fields, is *not* realized as a density matrix in the Fock representation (unless $\exp(-\beta H)$ was already a trace-class operator before the cutoff was introduced). It can be realized as a vector state in some other representation by a construction of the Wightman type [e.g., 85, chapter 3]. (Usually in the literature the smeared fields are exponentiated to form bounded Weyl operators [cf. 1] and the GNS (Gel'fand–Naimark–Segal) construction is applied (see, e.g., [59] and the articles by B. Simon and P.J.M. Bongaarts in the same book).) This construction can be carried out even if $\exp(-\beta H)$ *is* of trace class, as when M is compact. In that case the thermal state is realized both as a vector state in the (reducible) GNS representation and as a density-matrix state in the (irreducible) Fock representation.

If the spectrum of K (and hence K_β) extends down to 0, the integrals corresponding to (2.43) and (2.50) may or may not be infrared-convergent, as we shall demonstrate presently. In any event, the argument we have just given breaks down in that case. It may be possible to obtain its conclusion – in the cases where G^β is defined – by direct analysis of the mode of convergence of the spectral functions as the cutoff is removed, but we have not attempted to do so. Instead, we shall investigate the formulas (2.43) and (2.50) directly, in various cases of interest. The most interesting cases are those associated with cosmological horizons (including those resulting from acceleration); they are treated in section 4. Here we take a look at the simpler cases of the massless scalar field in flat rectangles, whose vacuum states were described at the end of section 2.2.

First consider the case where M is all of $\mathbf{R}^n$ and $K = -\nabla^2 + m^2$. Then [the continuum analogue of] (2.50b) can be written

$$G^\beta(s, x, y) = (2\pi)^{-n}\beta^{-1} \sum_{j=-\infty}^{\infty} \exp(2\pi i js/\beta) \int d^n k \exp\{ik\cdot(x-y)\} \left[\left(\frac{2\pi j}{\beta}\right)^2 + k^2 + m^2\right]^{-1} . \quad (2.55)$$

The right-hand side can just as well be regarded as the $T = 0$ Schwinger function for $M = S^1 \times \mathbf{R}^{n-1}$, where S^1 has circumference β; we return to this observation shortly. One may also write

$$G^\beta(s, x, y) = \beta^{-1} \sum_{j=-\infty}^{\infty} \exp(2\pi i js/\beta)\, S_n\left(|x-y|; \frac{(2\pi j)^2}{\beta^2} + m^2\right) . \quad (2.56)$$

For $m = 0$, (2.56) exhibits the fact that the Green function is not defined unambiguously for $n = 1$ or 2 and that this is due solely to the constant mode with respect to s; cf. (2.32), etc. Correspondingly, the continuous analogues of (2.43) and (2.50) are infrared-divergent for $n = 1, 2$.

For the more general rectangles, G^β is given by a formula like (2.33a), except that k_0 is now a discrete variable taking on the values

$$2\pi n/\beta \qquad (n = 0, \pm 1, \ldots)$$

as in (2.50b). The argument following (2.33a) therefore shows that in the thermal case one has infrared divergence whenever J, the number of "continuous" dimensions, is equal to either 1 or 2. [From (2.50a), the integrand of the analogue of (2.33b) involves a factor

$$\sim k^{-1}(1 - e^{-\beta k})^{-1} \sim \beta^{-1}k^{-2} \qquad \text{as } k \downarrow 0 , \quad (2.57)$$

where

$$k \equiv \left(\sum_{j=0}^{J} k_j^2\right)^{1/2}$$

is the radial coordinate in the continuous part of the wave-vector space. This also shows that $J \geq 3$ is needed for convergence.] Thus there are no finite-temperature equilibrium two-point functions (with vanishing chemical potential) for the massless free field in *three*-dimensional (or two-dimensional) Minkowski space-time, nor for the four-dimensional massless field in an infinite slab with Neumann conditions on its parallel plane faces, etc. We emphasize that the problem with the thermal states of the massless field in space-time dimensions 3 and 2 is of precisely the same nature as the pathology of the vacuum state of the massless field in dimension 2, which is by now well understood. In general, the infrared behavior of a theory at finite temperature is analogous to the zero-temperature situation in one lower dimension [66; 26 and references therein.]

Remark: One formulation of the criterion for existence of thermal two-point functions is: *The* C_0^∞ *test functions must belong to the domain of the operator* $K^{-1/2}$. (For the vacuum state, the test functions need only belong to $\mathrm{dom}(K^{-1/4})$, as remarked at the end of section 2.2.) Infrared convergence, in each of various contexts, requires that the test functions belong to the quadratic-form domain of a certain operator $R(K)$; such formulas as (2.19), (2.24–25), (2.43), (2.50), when smeared with test functions, are the spectral representations of these quadratic forms. We have $R(\omega) \sim \omega^{-1}$ for $\beta = \infty$ and $R(\omega) \sim \omega^{-2}$ for finite β. Since multiplication by ω^2 is the spectral representation of K, this means that the crucial space is the form domain of $K^{-1/2}$ for $\beta = \infty$ and the form domain of K^{-1} for finite β. The extra square root comes in in passing from quadratic-form domains to operator domains.

Finally, we have noted above that the elliptic Green function (2.55) corresponding to a free field in $\mathbf{R}^n$ at temperature $1/\beta$ is the same, up to a relabeling of axes, as that corresponding to the *vacuum* state of the field when one of the *spatial* dimensions is periodic with circumference β [e.g., $L_n = \beta/2$ in (2.9)]. This observation extends to interacting fields ([57]; cf. "Nelson's symmetry" [83, chapter 6]). Presumably related is the fact that the formula for the energy density of black-body radiation of fields of various spins is essentially the same as the formula for the "Casimir" energy density in spatially finite universes [84, 38, 16, 17]. Cf. remarks of Candelas and Dowker [18].

2.5. Image sums

The representation

$$\mathscr{G}^\beta(z, x, y) = \sum_{N=-\infty}^{\infty} \mathscr{G}^\infty(z + \mathrm{i}N\beta, x, y) \tag{2.58}$$

has been exploited in much of the literature [e.g., 14, 34, 6]. It arises in the first instance from the idea that the elliptic Green-function equation (2.26) with periodic boundary conditions in s should be solvable by summing the contributions of "image charges" at $s = N\beta$ in the "unrolled" manifold $\mathbf{R} \times \mathrm{M}$:

$$G^\beta(s, x, y) = \sum_{N=-\infty}^{\infty} G^\infty(s + N\beta, x, y)\,. \tag{2.59}$$

Does the series (2.59) converge? If K is the Laplacian on $\mathbf{R}^2$, then G^∞ is the Green function of the Laplacian on $\mathbf{R}^3$, better known as the Coulomb potential. The terms in (2.59) therefore decrease only as $|N|^{-1}$, and the sum diverges. Note, however, that for this case we already determined that no thermal two-point function exists. (A classical Green function on the cylinder does exist [cf. (2.32) and following discussion], but it does not have the positivity property needed for the Wightman reconstruction, and it cannot be constructed as an image sum. Are these two properties linked in general?) If K is the Laplacian on $\mathbf{R}^J$, $J>2$, then $G^\infty(s,x,y)$ decreases like r^{1-J} $[r \equiv (|x-y|^2+s^2)^{1/2}]$, and the sum converges. We have convergence also for the *massive* free field in *any* $\mathbf{R}^n$, since G^∞ then is damped by a factor $\exp(-mr)$; this can be generalized to most operators of the form (2.4) on $\mathbf{R}^n$ that possess strictly positive spectrum [51].

A systematic study is more easily carried out in the spectral representation. We can write

$$\mathscr{G}^\beta(z,x,y) = \int \mathrm{d}E_\omega(x,y)\, \mathscr{G}^\beta(z,\omega) \qquad (\beta \le \infty)\,, \tag{2.60}$$

where

$$\mathscr{G}^\infty(z,\omega) = \begin{cases} G_+^\infty(z,\omega) = (2\omega)^{-1}\,\mathrm{e}^{-\mathrm{i}\omega z} & \text{for } s<0\,, \\ G_-^\infty(z,\omega) = (2\omega)^{-1}\,\mathrm{e}^{+\mathrm{i}\omega z} & \text{for } s>0\,, \end{cases} \tag{2.61}$$

and $\mathscr{G}^\beta(z,\omega)$ for finite β is a periodic function of period $\mathrm{i}\beta$, determined by either of the equations

$$\mathscr{G}^\beta(z,\omega) = G_+^\beta(z,\omega) = (2\omega)^{-1}\,(1-\mathrm{e}^{-\beta\omega})^{-1}\,(\mathrm{e}^{-\mathrm{i}\omega z} + \mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{e}^{-\beta\omega}) \qquad \text{for } -\beta<s<0\,, \tag{2.62a}$$

$$\mathscr{G}^\beta(z,\omega) = G_-^\beta(z,\omega) = (2\omega)^{-1}\,(1-\mathrm{e}^{-\beta\omega})^{-1}\,(\mathrm{e}^{\mathrm{i}\omega z} + \mathrm{e}^{-\mathrm{i}\omega z}\,\mathrm{e}^{-\beta\omega}) \qquad \text{for } 0<s<\beta\,. \tag{2.62b}$$

[See (2.23), (2.19–20), (2.46), (2.43). The spectral function $E_\lambda(x,y)$ is given by (2.22) when the spectrum is discrete, and it can be constructed more or less explicitly whenever the field equation can be solved by separation of variables.]

From (2.61) and (2.62b) we obtain for $0<\operatorname{Im} z<\beta$

$$\begin{aligned} \sum_{N=-\infty}^{\infty} \mathscr{G}^\infty(z+\mathrm{i}N\beta,\omega) &= (2\omega)^{-1}\Big[\mathrm{e}^{\mathrm{i}\omega z}\sum_{N=0}^{\infty}\mathrm{e}^{-N\beta\omega} + \mathrm{e}^{-\mathrm{i}\omega z}\sum_{M=1}^{\infty}\mathrm{e}^{-M\beta\omega}\Big] \\ &= (2\omega)^{-1}\big[\mathrm{e}^{\mathrm{i}\omega z}(1-\mathrm{e}^{-\beta\omega})^{-1} + \mathrm{e}^{-\mathrm{i}\omega z}\,\mathrm{e}^{-\beta\omega}(1-\mathrm{e}^{-\beta\omega})^{-1}\big] = \mathscr{G}^\beta(z,\omega)\,. \end{aligned}$$

Since the extreme members of this equation are periodic, we have

$$\mathscr{G}^\beta(z,\omega) = \sum_{N=-\infty}^{\infty} \mathscr{G}^\infty(z+\mathrm{i}N\beta,\omega) \tag{2.63}$$

(absolutely) for all z with $\operatorname{Im} z$ not an integral multiple of β. Across the lines $\operatorname{Im} z = N\beta$, $\mathscr{G}^\beta(z,\omega)$ is discontinuous, with jump equal to the spectral transform of the commutator (2.17),

$$-\mathrm{i}\omega^{-1}\sin\omega t\,. \tag{2.64}$$

To establish (2.58) it is necessary to justify an interchange of the integration-summation over ω in (2.60) with the summation over N in the integrand, (2.63). We restrict attention to $\operatorname{Im} z \ne N\beta$, so that the ω integrations themselves will converge in a classical (not merely distributional) sense. In view of the polynomial bound on the spectral function, we have no problem at the ultraviolet end, hence (2.58) is certainly true when the spectrum of K is strictly positive. When the spectrum extends down to 0, we have seen in section 2.4 that there are cases where (2.60) for $\beta < \infty$ is divergent, hence the left-hand side of (2.58) is undefined, although every term on the right-hand side is defined, since (2.60) for $\beta = \infty$ is convergent. Let us consider the contrary case: $\mathscr{G}^\beta$ exists and the infrared part of its spectral representation,

$$\int_0^\lambda \mathrm{d}E_\omega(x, y)\,(2\omega)^{-1}\,(1 - \mathrm{e}^{-\beta\omega})^{-1}\,(\mathrm{e}^{\mathrm{i}\omega z} + \mathrm{e}^{-\mathrm{i}\omega z}\,\mathrm{e}^{-\beta\omega})\,,$$

converges. (Here λ is an arbitrary positive number, and we take $0 < s < \beta$.) The factors $\mathrm{e}^{\pm\mathrm{i}\omega z}$ are not responsible for this convergence, so the infrared convergence in (2.60) is absolute and (2.58) follows. It is not clear whether the converse holds: Does convergence of the right-hand side of (2.58) guarantee that $\mathscr{G}^\beta$ exists in the sense discussed in section 2.4?

The representation (2.58) enables one to read off the analytic structure of $\mathscr{G}^\beta$ from that of $\mathscr{G}^\infty$; note that this leads again to fig. 2. The flat case considered before illustrates the fact that the Riemann surface of $\mathscr{G}^\beta$ has infinitely many sheets whenever $\mathscr{G}^\infty$ has more than one sheet.

One may reformulate (2.58) as

$$G^\beta_*(t, x, y) = \sum_{N=-\infty}^{\infty} G^\beta_*(t + \mathrm{i}N\beta, x, y)\,,$$

where the $*$ stands for F, + or −, but such equations are sure to be misleading to one who is not fully aware of the analytic continuations, branch cuts, and distributional limits implicitly involved.

Remark: The rightmost member of (2.62a) or (2.62b) defines an *entire* function, which is *not* periodic. The values of each such function outside its proper strip are irrelevant; its spectral transform may not even converge outside that strip. A more elementary example of this phenomenon is provided by the function

$$[\cosh(x + 2\pi z/\beta)]^{-1}\,,$$

which is clearly $\mathrm{i}\beta$-periodic in z. For $|\mathrm{Im}\, z| < \beta/4$, its Fourier transform is unambiguous, and given by

$$\frac{\mathrm{e}^{\mathrm{i}2\pi z p/\beta}}{\cosh \pi p/2}\,,$$

but the analytic continuation to arbitrary z of the latter function is *not* periodic in z.

3. General quantum statistical systems: The KMS condition

3.1. Derivation and significance of the KMS condition

Consider an *arbitrary* quantum-mechanical system with time-independent Hamiltonian H. If A is an observable, its time evolution in the Heisenberg picture is

$$A_t = \mathrm{e}^{\mathrm{i}tH} A\, \mathrm{e}^{-\mathrm{i}tH}\,. \qquad (3.1)$$

If $e^{-\beta H}$ (for some $\beta > 0$) is of trace class, one can define the equilibrium state of temperature $T = 1/\beta$:

$$\langle A \rangle_\beta \equiv Z^{-1}\,\mathrm{Tr}(e^{-\beta H} A)\,, \tag{3.2}$$

$$Z \equiv \mathrm{Tr}\, e^{-\beta H}\,. \tag{3.3}$$

In manipulating expressions of form (3.2), one makes extensive use of the facts that $\mathrm{Tr}(O_1 O_2) = \mathrm{Tr}(O_2 O_1)$ and that any two functions of H alone commute.

For two observables, A and B, we define

$$\begin{aligned} G_+^\beta(t, A, B) &\equiv \langle A_t B \rangle_\beta \\ &= Z^{-1}\,\mathrm{Tr}[e^{-\beta H}\, e^{itH} A\, e^{-itH} B] \\ &= Z^{-1}\,\mathrm{Tr}[e^{-\beta H} A\, e^{-itH} B\, e^{itH}] = \langle A B_{-t} \rangle_\beta\,. \end{aligned} \tag{3.4}$$

In fact, for any t_1 and t_2 such that $t_2 - t_1 = t$ we have

$$\langle A_{t_2} B_{t_1} \rangle_\beta = G_+^\beta(t, A, B)\,. \tag{3.5}$$

Similarly, we define

$$\begin{aligned} G_-^\beta(t, A, B) &= \langle B A_t \rangle_\beta \\ &= Z^{-1}\,\mathrm{Tr}[e^{-\beta H} B\, e^{itH} A\, e^{-itH}] \\ &= \langle B_{-t} A \rangle_\beta = \langle B_{t_1} A_{t_2} \rangle_\beta \qquad \text{if } t_2 - t_1 = t\,; \end{aligned} \tag{3.6}$$

thus

$$G_-^\beta(t, A, B) = G_+^\beta(-t, B, A)\,. \tag{3.7}$$

Remark: If A or B is unbounded, the foregoing expressions may not be well defined. In particular, $e^{-\beta H} A$ is no longer of trace class in general. (For example, A could be $e^{+\beta H}$.) Therefore, in general discussions one usually assumes A and B to be *bounded* operators. Nevertheless, the formalism is applicable to thermal two-point functions in a box, in spite of the fact that the field operators are unbounded. This hinges on the fact that operators like $e^{-\beta H} a_\nu^\dagger a_\nu$ are of trace class, as we have seen explicitly in section 2.3.

Equations (3.4) and (3.6) can be rewritten as

$$G_+^\beta(z, A, B) = Z^{-1}\,\mathrm{Tr}[e^{i(z+i\beta)H}\, A\, e^{-izH}\, B]\,, \tag{3.8a}$$

$$G_-^\beta(z, A, B) = Z^{-1}\,\mathrm{Tr}[B\, e^{izH}\, A\, e^{-i(z-i\beta)H}]\,. \tag{3.8b}$$

Now z can be interpreted as a complex variable. If $z = t + is$, then both exponents in (3.8a) have negative real parts if $-\beta < s < 0$; for (3.8b), the condition is $0 < s < \beta$. Therefore, these two formulas define holomorphic functions in those respective strips. $G_{\pm}^{\beta}(t, A, B)$ are their boundary values.

If $0 \leq \operatorname{Im} z \leq \beta$, then

$$G_{-}^{\beta}(z, A, B) = G_{+}^{\beta}(z - \mathrm{i}\beta, A, B) . \tag{3.9}$$

Indeed, replacing z by $z - \mathrm{i}\beta$ in (3.8a) and cyclically permuting the factors inside the trace yields (3.8b). For $z = t$, (3.9) can be formally written

$$\langle BA_t \rangle_\beta = \langle A_{t-\mathrm{i}\beta} B \rangle_\beta , \tag{3.10}$$

but neither its derivation nor its applications require that (3.1) define an *operator* A_z for nonreal z. (In general it does not.)

Condition (3.9) or (3.10) is called the *KMS condition*, after Kubo [69] and Martin and Schwinger [70]. It can be given a precise sense in terms of C*- and Von Neumann algebras and their states for systems for which $\operatorname{Tr} \mathrm{e}^{-\beta H}$ diverges (e.g., noninteracting spatially infinite systems). In this context it is now accepted as a *definition* of "thermal equilibrium at temperature $1/\beta$" following Haag, Hugenholtz and Winnink [52]; see also [53, 79, 81, 67]. For some systems (particularly many infinite, interacting systems at low temperatures), (3.10) does not uniquely determine the state. This can be physically interpreted as the existence of more than one thermodynamic phase at that temperature. For detailed expositions of this subject, see [59] and [13, chapter 5].

3.2. Periodicity

The KMS condition obviously is related to the periodicity property of thermal two-point functions, (2.49); nevertheless, (3.9) and (3.10) as they stand are not statements of periodicity. We aim here to clarify this relationship.

Two analytic functions, G_+ and G_-, have been defined in disjoint, adjacent strips, just as in (2.46). Moreover, (3.9) states that one of these is the translate of the other. Therefore, we are free to *define* a periodic function throughout the complex plane, with the possible exception of the lines $s \equiv \operatorname{Im} z = N\beta$, by

$$\mathscr{G}^{\beta}(z, A, B) = G_{-}^{\beta}(z, A, B) \qquad \text{for } 0 < s < \beta , \tag{3.11a}$$

$$\mathscr{G}^{\beta}(z, A, B) = G_{+}^{\beta}(z, A, B) \qquad \text{for } -\beta < s < 0 , \tag{3.11b}$$

and, in general,

$$\begin{aligned} \mathscr{G}^{\beta}(z, A, B) &= G_{+}^{\beta}(z - \mathrm{i}N\beta, A, B) \\ &= G_{-}^{\beta}(z - \mathrm{i}(N-1)\beta, A, B) \end{aligned} \tag{3.11c}$$

for an integer N appropriate to the strip wherein z lies. $\mathscr{G}$ satisfies (2.49). But this construction does not have terribly much content in the most general case: $\mathscr{G}_\beta$ is not in general analytic on the real axis, and hence the functions defined in the various strips are *unrelated* except for the periodicity which has been imposed by fiat.

Suppose, however, that $A_t B = BA_t$ for all t in some open interval I of the real axis. Then the boundary values of $G_\pm^\beta(z, A, B)$ coincide on I, and just as in section 2 one can conclude by the edge-of-the-wedge theorem that $G_\pm^\beta$ are restrictions of a *single* holomorphic, periodic function, $\mathscr{G}^\beta(z, A, B)$, defined in a *connected* region making up all of the complex plane except parts of the lines $s = N\beta$. (In fact, in this case of bounded observables the desired result follows from the more elementary theorem of Painlevé [85].)

If $A_t B + BA_t = 0$ on I, one can modify the definition of $\mathscr{G}^\beta$ so that

$$\mathscr{G}^\beta(z, A, B) = -G_-^\beta(z, A, B) \qquad \text{for } 0 < s < \beta ,$$

obtaining an analytic function which is *antiperiodic*:

$$\mathscr{G}^\beta(z + \mathrm{i}\beta, A, B) = -\mathscr{G}^\beta(z, A, B)$$

(hence periodic with period 2β). In particular, on this basis the theory of thermal states of Fermi fields can be built up in parallel to section 2.

In relativistic field theories, the commutativity (or anticommutativity) of fields or observables at spacelike separations thus allows the construction of master holomorphic functions embodying the KMS condition as a periodicity property in imaginary time, as we demonstrated in detail for noninteracting scalar fields in section 2. This procedure does *not* apply to nonrelativistic quantum theory in the formalism of second quantization, since there the field operators at different times cannot be expected to commute anywhere; one has for *all* x and y the situation found in the relativistic case only when $x = y$. Thus the nonrelativistic case, which historically was studied earlier, is, in a sense, more complicated then the relativistic case! Understandably, much writing about finite temperature in relativistic theories has been based on transcriptions of results and arguments already known for the nonrelativistic case. The simple, powerful analytic structure of the two-point functions in the relativistic case has thereby been obscured in the literature. It is instructive to develop the nonrelativistic theory of thermal Green functions as far as possible along the lines adopted here for relativistic fields; we do that, sketchily, in appendix A.

3.3. Symmetries

We have previously emphasized that the periodicity condition (2.49) usually involves "jumping over a cut"; that is, it relates values of $\mathscr{G}^\beta$ at points in two different strips of holomorphy. On the other hand, combining (3.9) and (3.7) one arrives at

$$G_+^\beta(t - \mathrm{i}\beta, A, B) = G_+^\beta(-t, B, A) . \tag{3.12}$$

This equation relates boundary values of functions within the *same* strip; on the left side a limit as z approaches the bottom boundary of the strip from above is understood, while on the right, z approaches the top boundary from below. Of course, because of the interchange of A and B we are dealing here with two different functions; also, a reflection in t is involved. In some circumstances, various symmetries will eliminate one or the other of these "twists". These relationships are potentially useful – and also a potential source of confusion. Therefore, we shall dwell upon them awhile.

We return to the two-point function of a charged scalar field with equation of motion (2.8), (2.4).

For simplicity we assume no boundary conditions are necessary. (Many of the conclusions will extend to interacting scalar fields with the same symmetries.) A system of charged particles in an external magnetostatic potential A_j (but no electrostatic potential) ought to be invariant under CT, an operation of simultaneous charge conjugation and time reversal. If the magnetostatic potential vanishes, the system should also be invariant under C and T separately (even if the static gravitational potential, g_{jk}, and scalar (masslike) potential, V, are still present). In the field theory the C operator interchanges ϕ and $\phi^\dagger$. Recall also that any symmetry which includes time reversal involves a complex conjugation of matrix elements, as well as reflection of t.

An equilibrium state (with vanishing chemical potential) should exhibit the indicated symmetries, be it at zero or finite temperature. We shall now show that the systems considered by us indeed do this. The identity expressing CT invariance is

$$CT: \qquad \langle \phi(t, x)\, \phi^\dagger(0, y)\rangle = \langle \phi^\dagger(-t, x)\, \phi(0, y)\rangle^* . \tag{3.13a}$$

The right-hand side equals $\langle \phi^\dagger(0, y)\, \phi(-t, x)\rangle$. So, using the definitions (2.19–20) and (2.39–40), we rewrite (3.13a) as

$$CT: \qquad G_+(t, x, y) = G_-(-t, x, y) . \tag{3.13b}$$

From the explicit expressions (2.19–20) and (2.43), one sees that (3.13b) is satisfied by the vacuum and thermal two-point functions.

Charge-conjugation invariance by itself is

$$C: \qquad \langle \phi(t, x)\, \phi^\dagger(0, y)\rangle = \langle \phi^\dagger(t, x)\, \phi(0, y)\rangle . \tag{3.14a}$$

(For a neutral field this is a tautology.) Using time-translation invariance, we rewrite the right-hand side as $\langle \phi^\dagger(0, x)\, \phi(-t, y)\rangle$. It follows that in the G notation, the condition is

$$C: \qquad G_+(t, x, y) = G_-(-t, y, x) . \tag{3.14b}$$

Finally, for time reversal we have

$$T: \qquad \langle \phi(t, x)\, \phi^\dagger(0, y)\rangle = \langle \phi(-t, x)\, \phi^\dagger(0, y)\rangle^* ; \tag{3.15a}$$

after manipulations similar to the foregoing it becomes

$$T: \qquad G_+(t, x, y) = G_+(t, y, x) . \tag{3.15b}$$

(The same is true of G_-.) Now note that in the absence of the magnetic field, the complex conjugate of any eigenfunction of K (2.4) is also an eigenfunction with the same eigenvalue. Thus $\{\psi_j(x)\}_{\omega_j=\omega}$ and $\{\psi_j(x)^*\}_{\omega_j=\omega}$ are equally valid orthonormal bases for the ω-eigenspace, so the integral kernel of the projection operator onto that subspace can be written in either of the alternative forms

$$\sum_{\omega_j=\omega} \psi_j(x)\, \psi_j(y)^* = \sum_{\omega_j=\omega} \psi_j(x)^*\, \psi_j(y) . \tag{3.16}$$

This allows (3.14b) and (3.15b) to be verified on the explicit expressions (2.19–20) and (2.43). (This argument remains valid for continuous spectrum, *mutatis mutandis*.)

Remark: Were we not interested in covering the case with an external magnetic field, we would have written the basic field expansion as

$$\phi(t, x) = \sum_{\nu=1}^{\infty} (2\omega_\nu)^{-1/2} [\psi_\nu(x) \exp(-i\omega_\nu t)\, a_\nu + \psi_\nu(x)^* \exp(i\omega_\nu t)\, b_\nu^\dagger] \tag{3.17}$$

instead of (2.15). (Of course, by so doing one changes the definition of b_ν.) That is the only sensible formalism for treating the Hermitian field, where one wants to have $b_\nu^\dagger = a_\nu^\dagger$. In that formalism it is C, rather than CT, invariance which is immediately manifest without appeal to (3.16).

Now consider the consequences of combining these symmetries with the KMS condition, (3.9). In the present situation the latter becomes (on the boundary)

$$G_+^\beta(t - i\beta, x, y) = G_-^\beta(t, x, y)\,. \tag{3.18}$$

[Use the definitions (3.8) with $A = \phi(0, x)$, $B = \phi^\dagger(0, y)$.] If (3.13b), CT invariance holds, then

$$G_+^\beta(t - i\beta, x, y) = G_+^\beta(-t, x, y)\,. \tag{3.19}$$

If C holds, then

$$G_+^\beta(t - i\beta, x, y) = G_+^\beta(-t, y, x)\,. \tag{3.20}$$

If T holds, then

$$G_+^\beta(t - i\beta, x, y) = G_-^\beta(t, y, x)\,. \tag{3.21}$$

We can define a two-point function for $\phi^\dagger$ precisely analogous to G_+^β for ϕ:

$$H_+^\beta(t, x, y) \equiv Z^{-1}\, \mathrm{Tr}[e^{-\beta H} \phi^\dagger(t, x)\, \phi(0, y)] = G_-^\beta(-t, y, x) \tag{3.22}$$

(with a similar definition for $\beta = \infty$). Then (3.18) can be rewritten

$$G_+^\beta(t - i\beta, x, y) = H_+^\beta(-t, y, x)\,. \tag{3.23}$$

Similarly, if T holds, we have

$$G_+^\beta(t - i\beta, x, y) = H_+^\beta(-t, x, y)\,. \tag{3.24}$$

The periodicity condition (2.49) equates the value of $\mathcal{G}$ at the point A in fig. 3 to the value at point B (in the next strip). From one point of view, (3.18) is merely a restatement of that fact – since $G_\pm^\beta$ are restrictions of $\mathcal{G}$ to adjacent strips, as indicated in (2.46) – and (3.21) is a similar relation with an exchange of x and y. On the other hand, (3.21) is rewritten in (3.24) as an equality between the value of one two-point function at A and the value of a different one at C (in the same strip but with time

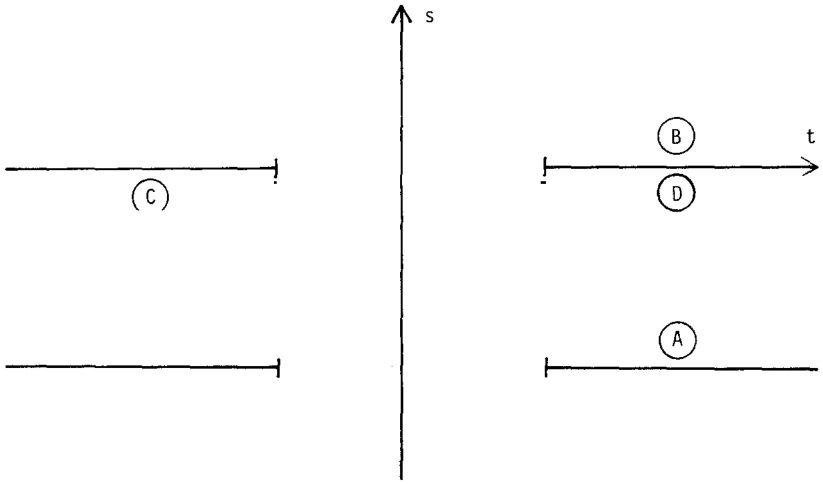

Fig. 3.

reversed). If CT holds, we have a relation (3.19) between A and C for the *same* two-point function. The relations (3.23) and (3.20) are like these but with the additional exchange of x and y.

In no case do we have an equation relating points A and D (*same* strip, same sign of t) – unless $\mathcal{G}$ happens to be continuous across the axis there, so that its values at B and D are equal. Such is of course the case when $|t| < d(x, y)$; it can also happen when $|t| > d(x, y)$ if the field satisfies Hugyens's principle, so that the commutator vanishes at timelike separations as well as spacelike ones. Example: the massless free scalar field in Minkowski space-time of even dimension greater than or equal to 4. See also [39, pp. 221–227] and references therein. In curved spaces, polelike singularities may be superimposed on the cut, because two points may be connected by both timelike and null geodesics [54, esp. caption to fig. 4].

3.4. Ordering in imaginary time

One occasionally encounters the misconception that working with the time-ordered field product [see (2.28) and (2.52)] instead of the ordinary product is responsible for eliminating the "twist" in the KMS condition and converting it into a simple statement of periodicity. This is entirely false: As we have seen, the time-ordered (Feynman) two-point function arises from the same analytic function, $\mathcal{G}^\beta$, that yields the ordinary (Wightman) two-point function upon choosing a different direction of approach to the cuts. We have periodicity in moving from A to B (fig. 3) independently of any time ordering; furthermore, for a fixed sign of t, G_{F} coincides with either G_+ or G_- and we do *not* have equality in moving from A to D (except in very special cases, as mentioned at the end of section 3.3). (Much of the confusion is due to lack of recognition of the existence of the cuts in the earliest papers on thermal Green functions in connection with black holes.)

What *is* relevant to the KMS condition is ordering in *imaginary* time [61, chapter 1]. Let $z = z_2 - z_1$, $s = \mathrm{Im}\, z$, $|s| < \beta$. In the notation of section 3.1 define

$$G_{\mathrm{F}}^\beta(z, A, B) \equiv \begin{cases} G_+^\beta(z, A, B) & \text{if } s < 0\,, \\ G_-^\beta(z, A, B) & \text{if } s > 0\,. \end{cases} \tag{3.25}$$

Then, formally,

$$G_{\mathrm{F}}^{\beta}(z, A, B) = \langle \mathcal{T}_{\vartheta}(A_{z_2} B_{z_1}) \rangle_{\beta} , \tag{3.26}$$

$$\mathcal{T}_{\vartheta}(A_{z_2} B_{z_1}) \equiv \begin{cases} A_{z_2} B_{z_1} & \text{if } s_2 < s_1 , \\ B_{z_1} A_{z_2} & \text{if } s_1 < s_2 . \end{cases} \tag{3.27}$$

(Note: The *smaller* value of s appears on the left operator.) From (3.11) we see that

$$G_{\mathrm{F}}^{\beta}(z, A, B) = \mathcal{G}^{\beta}(z, A, B) \qquad \text{for } |s| < \beta \tag{3.28}$$

– that is, throughout the two central strips in fig. 2. Thus

$$G_{\mathrm{F}}^{\beta}(z - \mathrm{i}\beta, A, B) = G_{\mathrm{F}}^{\beta}(z, A, B) \tag{3.29}$$

for $0 < s < \beta$ (and also at the boundaries, if the limits are taken properly).

The definition (3.27) can be extended to points where $\operatorname{Im} z = 0$, $\operatorname{Re} z \neq 0$ by putting the operator with the *larger* value of $\operatorname{Re} z$ on the left. Thus one recovers the usual $\mathcal{T}$ product by approaching the real axis from the directions indicated in fig. 1. This extension to equal imaginary times seems to us to be an arbitrary convention which is not particularly important, at least in the relativistic context. See also appendix A.

4. Horizons

4.1. *Hyperbolic polar coordinates (the Kruskal–Rindler transformation)*

A body at rest in the static gravitational field of a spherically symmetric mass must undergo a *constant acceleration* to avoid being pulled into the center of attraction. A static gravitational field is therefore a physical situation more closely analogous, in some ways, to flat space as experienced by a uniformly accelerated observer than to flat space as experienced by an inertial observer. The path of a uniformly accelerated observer through Minkowski space-time is a hyperbola. If this motion is continued for all time, there is an associated null (lightlike) hypersurface, the *future horizon*, beyond which lies a region of space-time from which signals can never reach the observer, and a *past horizon* bounding a region to which the observer could never have sent signals (see fig. 4). Together these are called simply "the horizon". A black hole, almost by definition, is a gravitational field in which a pair of null hypersurfaces exists which forms a horizon for every external static observer. (By "black hole" we shall mean here the maximal analytic extension of the vacuum Schwarzschild solution (4.9), rather than a nonstatic configuration containing a collapsing star.)

In the cases we shall discuss, the region R of space-time exterior to the horizon is static, hence has a time-translation symmetry group. The crucial point is that *in the vicinity of the horizon, these symmetries have the geometrical character of boosts* (homogeneous Lorentz transformations) *rather than ordinary time translations*. Indeed, in the case of uniform acceleration in flat space, they *are* a one-parameter group of boosts. A coordinate system manifesting the static nature of the region R must become singular on the horizon. The association of the horizon with finite temperature in quantum field theory

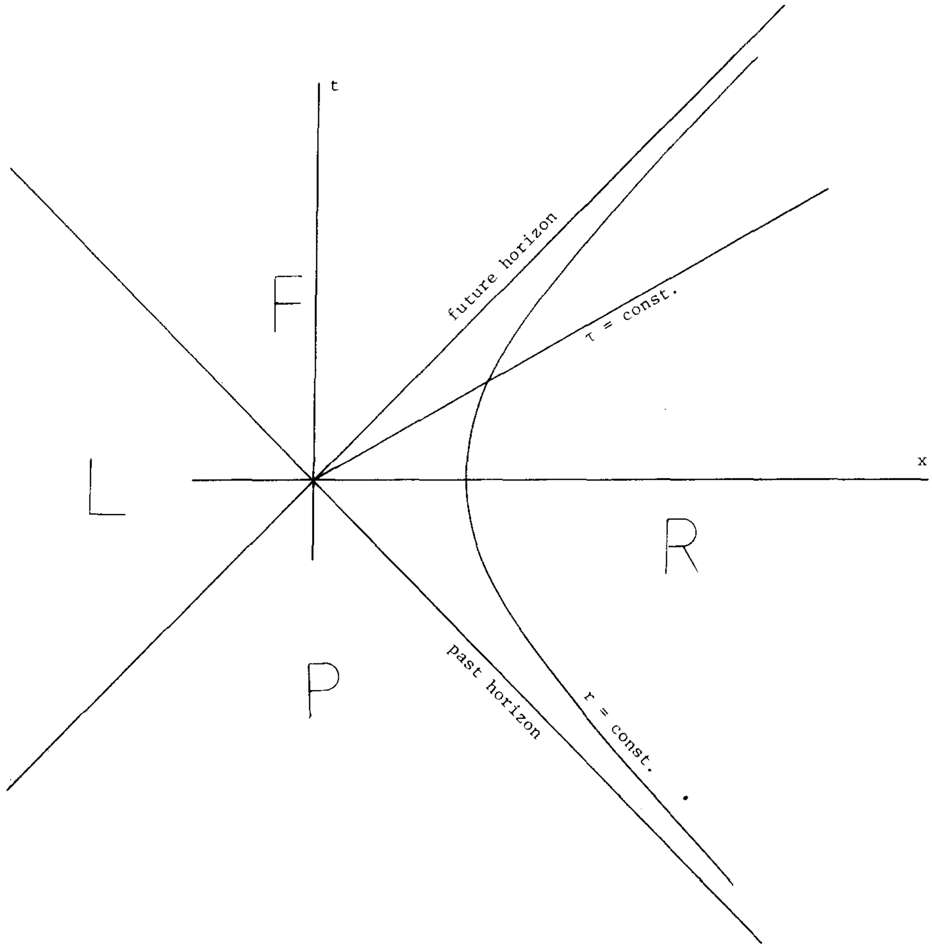

Fig. 4.

is closely connected with the mathematical nature of a transformation to a coordinate system which is regular on the horizon [68, 76, 42].

Consider first Minkowski space-time, of arbitrary dimension $n+1 \geq 2$. Let $x \equiv x^n$ be a Cartesian coordinate in the direction of the acceleration, and define new coordinates (τ, r) by

$$t = r \sinh \tau\,, \qquad x = r \cosh \tau\,. \tag{4.1}$$

Then the flat metric transforms this way:

$$-\mathrm{d}s^2 \equiv -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}\Omega^2 = -r^2\,\mathrm{d}\tau^2 + \mathrm{d}r^2 + \mathrm{d}\Omega^2 \tag{4.2}$$

where

$$\mathrm{d}\Omega^2 \equiv \sum_{j=1}^{n-1} (\mathrm{d}x^j)^2 \tag{4.3}$$

is the Euclidean metric of the transverse dimensions (if any). With the range $-\infty<\tau<\infty$, $0<r<\infty$, the coordinates cover the wedge-shaped region R ("Rindler space") defined by $|t|<x^n$; the lines parametrized by τ, with r and $\boldsymbol{x}_\perp \equiv (x^1, \ldots, x^{n-1})$ constant, are hyperbolic paths of constant acceleration (fig. 4).

The scalar field in R may be quantized directly by separation of variables in the coordinate system $(\tau, r, \boldsymbol{x}_\perp)$ [41, 12, 19, 15]. In the notation of section 2.1, we have here

$$\mathrm{M}=[0,\infty)\times\mathbf{R}^{n-1}\,, \qquad \mathcal{M}=\mathbf{R}\times\mathrm{M}=\mathrm{R}\,. \tag{4.4}$$

Since we wish to identify the field in R with the field in Minkowski space, it is convenient not to make the conformal transformation (2.12). Therefore, we will be working with the operator $\tilde{K}$ of (2.11), which has the form

$$\tilde{K}=-r^2\partial_r^2 - r\partial_r - r^2\nabla_\perp^2 + r^2m^2 \tag{4.5}$$

($\nabla_\perp^2 \equiv$ Laplacian with respect to the transverse dimensions, if any). Otherwise, section 2 applies unchanged.

We shall now show that the Rindler two-point function

$$\mathscr{G}_{\mathrm{R}}^{2\pi}(\zeta,(r_x,\boldsymbol{x}_\perp),(r_y,\boldsymbol{y}_\perp)) \qquad (\zeta\equiv\tau+\mathrm{i}\sigma) \tag{4.6}$$

is equal (after composition with the coordinate transformation) to the Minkowski two-point function

$$\mathscr{G}_{\mathrm{M}}^{\infty}(z,\boldsymbol{x},\boldsymbol{y}) \qquad (z\equiv t+\mathrm{i}s)\,,$$

which was studied in section 2.2. There we saw that

$$\mathscr{G}_{\mathrm{M}}^{\infty}(z,\boldsymbol{x},\boldsymbol{y}) = F(-z^2+|\boldsymbol{x}-\boldsymbol{y}|^2)\,,$$

where $F(w)$ is a function holomorphic except at $w=0$. Restricting the space-time points to the Rindler wedge and using (4.1) yields

$$\mathscr{G}_{\mathrm{M}}^{\infty}(t_2-t_1,\boldsymbol{x},\boldsymbol{y}) = F(r_x^2+r_y^2-2r_xr_y\cosh(\tau_2-\tau_1)+|\boldsymbol{x}_\perp-\boldsymbol{y}_\perp|^2)\,. \tag{4.7}$$

To see that (4.7) is the same as (4.6), note first of all that the right-hand side of (4.7) with $\tau_2-\tau_1$ replaced by ζ has the correct analyticity and periodicity structure in ζ; cf. fig. 2. Therefore it suffices to show that its restriction to $\zeta=\mathrm{i}(\sigma_2-\sigma_1)$ (σ_j real) is the kernel of the operator

$$(-\partial^2/\partial\sigma^2+\tilde{K})^{-1}$$

on $\mathrm{L}^2(\mathrm{S}^1\times\mathrm{M};\, r^{-1}\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}\Omega)$, where S^1 is a circle of circumference 2π. But this restriction is just $G_{\mathrm{M}}^{\infty}(s_2-s_1,\boldsymbol{x},\boldsymbol{y})$ written in terms of polar coordinates defined by

$$s=r\sin\sigma\,, \qquad x=r\cos\sigma\,. \tag{4.8}$$

Thus, the desired equation (2.26), which reads here

$$\left[-\frac{\partial^2}{\partial\sigma_2^2}+\tilde{K}_{(x)}\right] G_{\mathrm{R}}^{2\pi}(\sigma_2-\sigma_1,(r_x,\boldsymbol{x}_\perp),(r_y,\boldsymbol{y}_\perp))=\delta(\sigma_2-\sigma_1)\,\delta(r_x-r_y)\,\delta(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)\,r_y\,,$$

is simply a transcription to polar coordinates on the (s, x) plane of the equation

$$\left[-\frac{\partial^2}{\partial s_2^2}-\nabla^2_{(x)}+m^2\right] G_{\mathrm{M}}^{\infty}(s_2-s_1,\boldsymbol{x},\boldsymbol{y})=\delta(s_2-s_1)\,\delta(\boldsymbol{x}-\boldsymbol{y})\,,$$

and the conclusion follows.

As a result, the Minkowski vacuum state may be viewed, on restriction to R, as the Rindler thermal state with $\beta=2\pi$. When one takes the normalization of the Rindler time coordinate into account, one sees that the effective Rindler temperature at a point $(\tau_0, r_0, \boldsymbol{x}_0)$ in R is proportional to the acceleration corresponding to the hyperbola $(\tau, r_0, \boldsymbol{x}_0)$ through that point – i.e., to r_0^{-1}. For further discussion of physical implications see Sciama et al. [78] and Bell and Leinaas [4].

For comparison with the Schwarzschild case later, it is useful to look at the foregoing development from this angle: The vacuum two-point function, $G_{\mathrm{M}+}^{\infty}(t, \boldsymbol{x}, \boldsymbol{y})$, of a free scalar field in Minkowski space has an analytic continuation, $\mathscr{G}^{\infty}(z, \boldsymbol{x}, \boldsymbol{y})$, to complex values of $z\equiv t+\mathrm{i}s$, which is holomorphic along the whole s axis if $\boldsymbol{x}\neq\boldsymbol{y}$. Such a holomorphic function is necessarily a 2π-periodic (and holomorphic) function of σ when polar coordinates are introduced by (4.4). (It is also possible to interpolate between real ζ and imaginary ζ, at the cost of allowing the Minkowski spatial coordinate, x^n-y^n, to become complex. One can check that, for fixed positive r, such points remain inside the tube where the vacuum two-point function is analytic [cf. 85, esp. theorems 2.8 and 3.5].)

Historical remarks: The first inkling that acceleration is associated with temperature in quantum field theory was obtained by Davies [25], who applied Hawking's derivation of black-hole radiance [55] to a reflecting barrier in flat space. The landmark paper on the subject, by Unruh [88], uses the methods which are the subject of our section 4.2. The approach through the KMS condition on the two-point function, which we have described here, was developed by many authors, including Dowker [33, 34], Christensen and Duff [23], and DeWitt [28], in response to analogous developments in the theory of black holes – to which we turn next. Meanwhile, Bisognano and Wichmann [8, 9] proved a related theorem in axiomatic field theory, whose relevance to the horizon problem has been stressed by Sewell [80, 82] and Kay [64, 65].

Now consider the (4-dimensional) Schwarzschild metric, which describes a nonrotating, uncharged black hole of mass M:

$$-\mathrm{d}s^2=-(1-2M/r)\,\mathrm{d}t^{*2}+(1-2M/r)^{-1}\,\mathrm{d}r^2+r^2\,\mathrm{d}\Omega^2\,, \tag{4.9}$$

where now

$$\mathrm{d}\Omega^2=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\,, \tag{4.10}$$

the usual angular element in spherical coordinates. The horizon is at $r=2M$, and the range of relevance of (4.9), analogous to R, is $2M<r<\infty$, $-\infty<t^*<\infty$. If

$$r^*=r+2M\ln(r/2M-1) \tag{4.11}$$

(the Regge–Wheeler, or tortoise, coordinate), so that

$$-\infty < r^* < \infty\,, \qquad \mathrm{d}r^*/\mathrm{d}r = (1-2M/r)^{-1}\,, \tag{4.12}$$

then

$$-\mathrm{d}s^2 = (1-2M/r)(-\mathrm{d}t^{*2} + \mathrm{d}r^{*2}) + r^2\,\mathrm{d}\Omega^2\,. \tag{4.13}$$

If we introduce coordinates (t, x) by (4.1) with

$$\tau \equiv t^*/4M \tag{4.14a}$$

and r in (4.1) replaced by

$$r' \equiv 2M\,\mathrm{e}^{\rho} \equiv 2M\,\mathrm{e}^{r^*/4M} = [2M(r-2M)]^{1/2}\,\mathrm{e}^{r/4M}\,, \tag{4.14b}$$

then we get

$$-\mathrm{d}s^2 = \frac{8M}{r}\,\mathrm{e}^{-r/2M}(-\mathrm{d}t^2 + \mathrm{d}x^2) + r^2\,\mathrm{d}\Omega^2\,, \tag{4.15}$$

where r is defined implicitly by (4.11). The original exterior Schwarzschild region corresponds to $x > |t|$, but the metric (4.15) extends analytically to a much larger region of the (t, x) plane. Figure 4 applies without change, except that the regions F and P now terminate (at a true singularity of the geometry) at hyperbolic boundaries, $t^2 - x^2 = 4M^2$ (corresponding to $r = 0$ in the conventional continuation of (4.9) across the horizon). This is the famous Kruskal extension [68].

Since the Kruskal metric (4.15) is not static (r depends on t as well as x), a quantum field propagating in this background does not have a vacuum state in the usual sense. So the previous discussion of the vacuum in Minkowski space cannot be carried over intact to the present situation. Nevertheless, like Minkowski space, Kruskal space does have an analytic extension to a complex manifold, the imaginary-time section of which is a real manifold of positive definite metric where the hyperbolic time coordinate is transmuted into an angular coordinate. The metric of this space is

$$\frac{8M}{r}\,\mathrm{e}^{-r/2M}(\mathrm{d}s^2 + \mathrm{d}x^2) + r^2\,\mathrm{d}\Omega^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}s^{*2} + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{4.16}$$

where, in analogy to t^* and z,

$$s^* \equiv 4M\sigma\,, \qquad \zeta \equiv \tau + \mathrm{i}\sigma\,; \tag{4.17}$$

and another way of stating the point is that, having arrived at the right-hand side of (4.16) by formal analytic continuation of (4.9), one finds that the only assignment of a *period*, β, to s^* which *eliminates* the singularity on the axis ($r = 2M$) is $\beta = 8\pi M$ [48, 50, 56].

It is then natural to consider a state whose two-point function (or Feynman propagator) analytically continues to this imaginary-time manifold; Hartle and Hawking [54] gave an argument for assuming this to be the case, based on path integrals for relativistic particles. Such a state will satisfy the KMS condition on the original Schwarzschild space-time, with temperature $T_M \equiv (8\pi M)^{-1}$ – the same temperature associated by Hawking [55] with a star collapsing to form a black hole of mass M. Gibbons and Perry [49, 50] continued this analysis and identified the Hartle–Hawking propagator with that appropriate to a black hole in equilibrium with a gas (of field quanta) of temperature T_M at infinity (cf. section 4.2). Gibbons and Perry (see also [56]) gave a qualitative, perturbative argument that any interacting field must also have a natural equilibrium state with the black hole at this temperature; Sewell [80, 82] made a rigorous, axiomatic study of this question, starting from technical assumptions whose validity will merit future investigation.

Similar observations have been made about de Sitter space (a space-time of constant, nonzero curvature) by Figari, Höegh-Krohn and Nappi [36], Gibbons and Hawking [47], Dowker [34], and others. In this case the temperature turns out to be proportional to the curvature, and the time variable involved in the KMS condition is the "private" proper time of any inertial observer, extended to a Fermi normal coordinate system – which necessarily terminates in a horizon.

We should also mention a similar construction by analytic continuation in cosmological – nonstatic – contexts [20–22]. In this case the time continues to the radial, rather than the angular, coordinate.

4.2. The doubling of Fock space

Simultaneously with the work of Hartle, Hawking, Gibbons and Perry (section 4.1), the connection between horizons and temperature was being worked out by an alternative method [88, 60, 42] (see also [63–65, 30]). Although it also involves analytic continuation, this approach concentrates attention more on the physical (real) space-time. We summarize it here briefly.

In either Minkowski or Schwarzschild space-time (or any static model with a horizon), the scalar field can be quantized in the region R in the standard way [41, 10, 12]. The field expansion has the form (2.15) (or rather its continuum analogue) for certain eigenfunctions ψ_ν, whose explicit form will not concern us till section 4.3. There is an identical construction for the region L, symmetrical to R on the other side of the horizon (see fig. 4), with basis functions $\tilde{\psi}_\nu$, say.

A particular vacuum state and associated Fock space are implied by writing down (2.15). What is important here is the splitting of the field into a positive-frequency, or annihilation, part and a negative-frequency, or creation, part, not the particular basis $\{\psi_\nu\}$ employed. Unruh enquired how the positive-frequency normal modes in R and L are related to the positive-frequency modes in the standard quantization of the free field in Minkowski space. The latter are plane waves; Unruh noted that they can be characterized by their property of holomorphy in the lower half-planes of the complex variables

$$V \equiv t + x\,, \qquad U \equiv t - x\,. \tag{4.18}$$

It turns out (see the cited references for details) that this property is shared by uniquely determined (up to phase) linear combinations of pairs of modes, one from R and one from L:

$$\begin{aligned}\phi_\nu(t, x) &\equiv [2\sinh(\pi\omega_\nu)]^{-1/2} \exp(-\mathrm{i}\omega_\nu \tau)\, [\exp(\pi\omega_\nu/2)\, \psi_\nu(r) + \exp(-\pi\omega_\nu/2)\tilde{\psi}^*_\nu(r)] \\ &= \exp(-\mathrm{i}\omega_\nu \tau)\, (\cosh\theta_\nu\, \psi_\nu + \sinh\theta_\nu\, \tilde{\psi}^*_\nu)\,,\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}\cosh\theta_\nu &\equiv (1 - \exp(-2\pi\omega_\nu))^{-1/2}\,, \\ \sinh\theta_\nu &\equiv \exp(-\pi\omega_\nu)\cosh\theta_\nu\,.\end{aligned} \tag{4.20}$$

(We have written the coefficients in two different but equivalent forms to make contact with the notations of several references at once.) The basis change (4.19) induces a redefinition (Bogolubov transformation) of the annihilation and creation operators:

$$A_\nu = \cosh\theta_\nu\, a_\nu - \sinh\theta_\nu\, \tilde{b}_\nu^\dagger\,. \tag{4.21}$$

The A_ν are, by the analyticity argument, (continuum) linear combinations of the annihilation operators of the standard quantization.

From (4.21) it follows that the vacuum of the standard quantization is full of particles from the point of view of the original quantization in R. From the explicit form (4.20) of the coefficients, it follows that these particles form precisely a thermal gas with inverse temperature $\beta = 2\pi$. So the conclusion of section 4.1 is reproduced.

On the Schwarzschild–Kruskal manifold there is no "standard" quantization *a priori*. However, the geometry of the horizon (more precisely, of its projection into the (t, x) plane – the remaining coordinates playing no essential role) is the same as in Minkowski space. It therefore seems likely that the physically most natural quantization is defined by normal modes with the same analytic property as the flat-space plane waves. One therefore repeats the construction (4.19–21) in the black-hole context, and obtains a state which, relative to the original quantization in the static exterior Schwarzschild metric (4.9), is characterized by inverse temperature $\beta = 8\pi M$. (The extra factor $4M$ arises from a difference in conventional normalization of coordinates – cf. (4.17).) This state is, of course, the Hartle–Hawking–Gibbons–Perry equilibrium state previously discussed. (Unruh actually applied this construction to only half the modes to obtain a state in which the black hole is radiating into empty space.)

The analytic continuation assumed in this approach is more conservative than that in the other approach, because it is required only on the horizon, not in the whole space R. On the other hand, the statement that the constructed state comprises a thermal gas of quanta, although formally unimpeachable [88, 60, 78, 86, 63 (appendix 1)], glosses over the technicalities related to the continuous spectrum, which prevents the Hamiltonian of the system from being a trace-class operator (see remarks in sections 2.3 and 2.4, see also [89]). Perhaps the most efficient way to travel around this problem is to pass *a posteriori* to the imaginary-time manifold, verify the KMS condition for expectation values, and appeal to the GNS construction. (See also [30].)

Israel [60] noted that the construction (4.20–21) already existed in the literature of general quantum statistical mechanics. It appears in the work of Araki and Woods [1, section 4 and appendix 1] and was rediscovered by Takahashi and Umezawa [86], who called it "thermo field dynamics" and developed it as a computational tool. (See also [3, 71, 87 (esp. chapter 4), 73, 74, 63, 64, 72].) In this work the degrees of freedom associated with operators $\tilde{a}_\nu$, $\tilde{b}_\nu$ are a fictitious mathematical device, not associated with a second physical region L.

There is a relationship between this Araki–Woods construction and the KMS condition via what is now called the Tomita–Takesaki modular theory (for which see [59, sections 5–7]). This connection [which is summarized in Ojima's papers] plays an important role in the original paper of Haag, Hugenholtz and Winnink [52] on the KMS condition. The work of Bisognano and Wichman [8, 9] has a bearing on a geometrical realization of the modular structure in the case of (interacting) fields in (flat) Rindler space. (The modular conjugation becomes a reflection of R onto L.) The work of Sewell [82] and Kay [63, 64] is partly concerned with extending this analysis to the Schwarzschild horizon.

4.3. Infrared questions

The operator $\tilde{K}$ (4.5) of Rindler space has a spectrum which extends all the way down to $\omega^2 = 0$, even if the mass of the field is positive. The same is true of the analogous operator for Schwarzschild space. Therefore, it cannot be taken for granted that the various two-point functions actually exist in these important contexts. Kay [65, esp. theorem 4.5] has studied some essentially equivalent issues by operator-theoretic methods. In this section we investigate the existence of thermal states via eigenfunction expansions. For Rindler space our results are complete, including cases which Kay left open. For Schwarzschild space, we are unable yet to give a conclusive answer by our methods; since Kay handles this case without difficulty, we keep our remarks on it very brief.

As discussed in sections 2.2 and 2.4, what is needed is to show convergence of

$$\sum_\alpha \int_0 \omega^{-1} |\tilde{f}(\omega;\alpha)|^2 \, \mathrm{d}\omega \qquad \text{for the vacuum} \tag{4.22a}$$

and of

$$\sum_\alpha \int_0 \omega^{-2} |\tilde{f}(\omega;\alpha)|^2 \, \mathrm{d}\omega \qquad \text{for thermal states}\,, \tag{4.22b}$$

for arbitrary C_0^∞ functions f with eigenfunction transform

$$\tilde{f}(\omega;\alpha) \equiv \int_M \psi_{\omega\alpha}(x)^* \, f(x) \, \gamma(x)^{1/2} \, \mathrm{d}^n x\,. \tag{4.23}$$

Here ω^2 is the variable ranging over the spectrum of K, α is a schematic "auxiliary quantum number", and the eigenfunctions of K are normalized through

$$\int \psi_{\omega\alpha}(x)^* \, \psi_{\omega'\alpha'}(x) \, \gamma(x)^{1/2} \, \mathrm{d}^n x = \delta(\omega-\omega') \, \delta(\alpha-\alpha')\,. \tag{4.24}$$

We also recall that convergence of (4.22a) or (4.22b) is equivalent to

$$C_0^\infty \subset \operatorname{dom} K^{-1/4}\,, \qquad C_0^\infty \subset \operatorname{dom} K^{-1/2}\,, \tag{4.25}$$

respectively.

The physical interest of this issue is heightened by its close relationship to the phenomenon of Bose–Einstein condensation. Throughout this paper we have assumed that the chemical potential μ vanishes. In physical terms this amounts to assuming that the "particles" involved behave like photons or phonons. However, if they behave like, say, ^{4}He atoms, one should allow $\mu \neq 0$, and then the convergence or divergence of the integrals (4.22a, b) amounts to the occurrence or absence of Bose–Einstein condensation; see, e.g., [13, chapter 5].

In the Rindler case it is expedient to trade the variable r for a variable

$$\rho \equiv \ln r \qquad (-\infty < \rho < \infty)\,, \tag{4.26}$$

so that $\tilde{K}$ assumes the Schrödinger form

$$\tilde{K} = -\partial^2/\partial\rho^2 + (-\Delta_\perp + m^2)\,e^{2\rho} \tag{4.27}$$

and acts on the Hilbert space $L^2(M, d\rho\, d\Omega)$, where M is now simply $\mathbf{R}^n$. In particular, when $n = 1$ and $m = 0$ one has $\tilde{K} = -\partial^2/\partial\rho^2$, so that (4.23) amounts to Fourier transformation. Thus, both (4.22a) and (4.22b) diverge whenever $\int_{\mathbf{R}} f(\rho)\, d\rho \neq 0$.

On the other hand, in the cases $m > 0$, $n = 1$ and $m \geq 0$, $n > 1$ the eigenfunctions are [42, 78]

$$\psi_{\omega k}(\rho, \boldsymbol{x}_\perp) = c(\omega \sinh \pi\omega)^{1/2}\, K_{i\omega}(\kappa_k\, e^\rho) \exp(i\boldsymbol{k}\cdot \boldsymbol{x}_\perp) \tag{4.28}$$

where

$$\kappa_k \equiv (k^2 + m^2)^{1/2}\,. \tag{4.29}$$

From the integral representation

$$K_{i\omega}(x) = \int_0^\infty e^{-x\cosh t} \cos\omega t\, dt \qquad (x > 0) \tag{4.30}$$

it is obvious that

$$|K_{i\omega}(x)| \leq K_0(x) \qquad \text{for } x > 0\,. \tag{4.31}$$

Thus if we insert (4.28) into (4.23) and do the $\boldsymbol{x}_\perp$ integral we infer that

$$|\tilde{f}(\omega; \boldsymbol{k})| \leq c\omega \int_{\operatorname{supp} f} K_0(\kappa_k\, e^\rho)\, d\rho\,, \tag{4.32}$$

where c depends only on f. From this it easily follows that both integrals (4.22) converge. (To see this when $m = 0$ and $n > 1$, recall that $K_0(x) \sim \ln(1/x)$ as $x \to 0$.)

We conclude that in the Rindler case the two-point functions exist for $m > 0$, and also for $m = 0$ provided that at least one "infinite" transverse dimension, ensuring a continuous k spectrum, is present. In particular, G^β_{R+} exists for every $\beta \leq \infty$ in three-dimensional Rindler space-time, unlike the three-dimensional massless Minkowski case, where G^β_{M+} does not exist for $\beta < \infty$. For $\beta = 2\pi$ this result could have been predicted: $G^{2\pi}_{R+}$ is a restriction of G^∞_{M+}, which has no infrared pathology in three dimensions (cf. sections 4.1 and 2.2).

In the Schwarzschild case, it is well known (see, e.g., [27, section 5.1] for $m = 0$ and [37] for $m > 0$) that the analogues of the Bessel functions in (4.2c) are eigenfunctions of operators

$$K_l = -d^2/dr^{*2} + V_l(r^*) \tag{4.33}$$

where the potentials

$$V_l(r^*) \equiv \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + m^2\right) \tag{4.34}$$

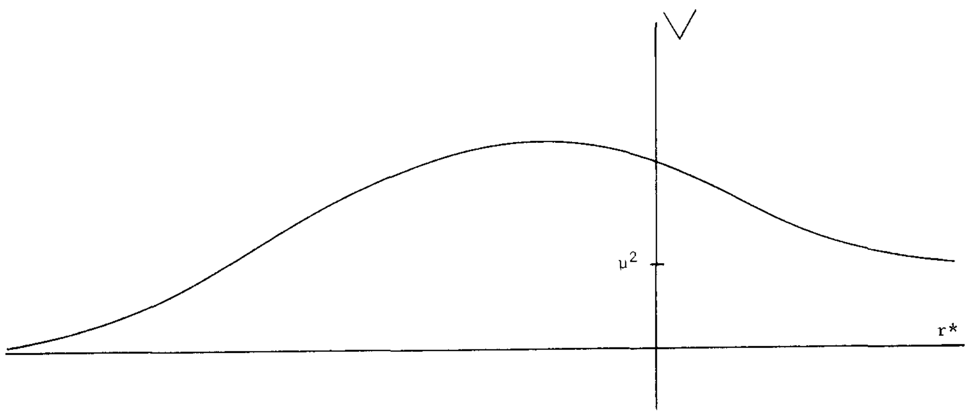

Fig. 5.

have the schematic behavior graphed in fig. 5. The question whether (4.23) holds is equivalent to the question whether $C_0^\infty(\mathbf{R})$ belongs to the domain of each K_l^α, $\alpha = -\frac{1}{4}$ or $-\frac{1}{2}$. Unfortunately, the Schrödinger equation for K_l cannot be solved in terms of standard special functions as in the Rindler case. One would expect, nevertheless, to find the answer to our question by inspecting the extensive lore on one-dimensional Schrödinger operators and their eigenfunctions; surprisingly, this appears not to be the case. It would suffice to prove a bound of the form

$$\sup_{r^* \in \mathrm{B}} |\psi_{\omega l}(r^*)| \le C_{\mathrm{B},l}\,\omega^\alpha\,, \tag{4.35}$$

$\omega \to 0$, $\alpha > \frac{1}{2}$, for arbitrary bounded subsets $\mathrm{B} \subset \mathbf{R}$, where $\psi_{\omega l}$ are the eigenfunctions of K_l with the standard normalization at infinity (guaranteeing (4.24)). Of course, it is physically obvious that $\psi_{\omega l}(r^*)$ with r^* fixed vanishes when $\omega \to 0$: As the energy goes to zero, the turning point for a classical particle hitting the potential hill moves off to $-\infty$, and even a quantum particle will not penetrate far beyond the turning point. However, just how fast the wave function vanishes cannot be established by heuristics alone. One might hope that a rigorous asymptotic analysis of these eigenfunctions, guided by the analogy between the Schwarzschild and Rindler potentials near the horizon ($r^* \to -\infty$), would establish the desired behavior; but such a study is beyond the scope of the present paper.

4.4. Concluding remarks

The discovery in the mid-1970s of an unexpected relationship between geometrical horizons and thermal effects stimulated much speculation that a profound unification of gravitation, quantum theory, and thermodynamics was at hand. Further investigation clarified this mysterious relationship by fitting it into the respective frameworks of two characterizations of finite-temperature equilibrium states which were already part of ordinary (nongravitational) quantum statistical mechanics: the Araki–Woods construction of a double Fock space, and the Kubo–Martin–Schwinger condition of periodicity in imaginary time. The periodicity emerges automatically in the context of a horizon, because the natural static coordinate system for the problem is a hyperbolic polar coordinate system associated to the horizon. Likewise, the doubling of the Fock space is the unsurprising consequence of the division of physical space into two noncommunicating parts by the horizon [cf. 64].

On the other hand, within quantum statistical mechanics itself both the KMS condition and the Araki–Woods construction are simply mathematical facts, for which clear and direct physical motivations are hard to find. This problem is acknowledged in the literature of the subject:

> The mathematical structure of the representation, as we have constructed it, is rather suggestive. [The field] can be interpreted as the sum of an annihilation operator on the first space (or first kind of particle) and a creation operator on the second space (or second kind of particle). This immediately brings to mind the particle–antiparticle description in elementary particle physics. . . . However we do not understand the significance of these remarks. [1]

> Starting from the Gibbs ensemble it is evident that [the KMS] condition is satisfied but it is completely unclear that this condition alone should characterize equilibrium. Nevertheless, this is the case for a large class of [systems]. This rather surprising result is both of practical and conceptual utility. . . . [T]he KMS condition has a variety of characterizations which emphasize different physical features such as stability under perturbations and ergodicity in the form of asymptotic Abelianness. This clarifies to a large extent the nature of the equilibrium states even if it does not provide any profound explanation for their definition. [13]

So, although the gravitational or geometrical half of the connection is now manifest, at least for the "free" fields considered in this paper, a completely satisfying, intuitive understanding of the thermal nature of horizons remains elusive, even for this simple case.

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Appendix A. Nonrelativistic "free" fields and parabolic Green functions

We outline the analogue of the developments in section 2 for a nonrelativistic system of noninteracting bosons. We shall again consider a manifold M and an operator K given by (2.4). However, in the nonrelativistic context K represents the one-particle energy, and $V(x)$ an electrostatic potential. We shall again assume (2.5)–(2.7), with the additional restriction that the spectrum of K has a positive lower bound. For notational convenience we shall also assume that K has no continuous spectrum.

With these assumptions, our system can be described by a quantum field $\psi(t, x)$, satisfying

$$\mathrm{i}\partial\psi/\partial t = K\psi \tag{A1}$$

and canonical commutation relations. Specifically, we may take

$$\psi(t, x) = \sum_{\nu=1}^{\infty} \psi_\nu(x) \exp(-\mathrm{i}\omega_\nu^2 t)\, a_\nu \tag{A2}$$

which is the analogue of (2.15). We mention in passing that what follows has an obvious translation to the fermion case: One needs only to replace the boson annihilators in (A2) by fermion ones. (In contrast, for the Klein–Gordon field (2.15) this replacement does *not* lead to canonical anti-commutation relations. To describe noninteracting fermions relativistically one needs a different field, such as the Dirac field.)

In analogy with (2.19) and (2.20), one has vacuum two-point functions

$$\begin{aligned} G_+^\infty(t, x, y) &\equiv \langle 0|\psi(t_2, x)\, \psi^\dagger(t_1, y)|0\rangle \\ &= \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^* \exp(-\mathrm{i}\omega_\nu^2 t) \qquad (t \equiv t_2 - t_1) \end{aligned} \tag{A3}$$

and

$$G_-^\infty(t, x, y) \equiv \langle 0|\psi^\dagger(t_1, y)\, \psi(t_2, x)|0\rangle = 0\,. \tag{A4}$$

By repeating the *ab initio* calculation of the Gibbs two-point functions in sections 2.3 and 2.4 one finds the finite-temperature two-point functions

$$\begin{aligned} G_+^\beta(t, x, y) &\equiv \langle \psi(t_2, x)\, \psi^\dagger(t_1, y)\rangle_\beta \\ &= \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^* (1 - \exp(-\beta\omega_\nu^2))^{-1} \exp(-\mathrm{i}\omega_\nu^2 t)\,, \end{aligned} \tag{A5a}$$

$$\begin{aligned} G_-^\beta(t, x, y) &\equiv \langle \psi^\dagger(t_1, y)\, \psi(t_2, x)\rangle_\beta \\ &= \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^* (1 - \exp(-\beta\omega_\nu^2))^{-1} \exp(-\beta\omega_\nu^2) \exp(-\mathrm{i}\omega_\nu^2 t)\,, \end{aligned} \tag{A5b}$$

which clearly satisfy

$$G_+^\beta(t, x, y) - G_-^\beta(t, x, y) = [\psi(t, x), \psi^\dagger(0, y)] = G_+^\infty(t, x, y)\,. \tag{A6}$$

These equations are to be compared with (2.43) and (2.44).

Setting $z \equiv t + \mathrm{i}s$, we now write

$$G_\pm^\beta(z, x, y) = \sum_{\nu=1}^{\infty} \psi_\nu(x)\, \psi_\nu(y)^* (1 - \exp(-\beta\omega_\nu^2))^{-1}\, q_\pm^\beta(z) \tag{A7}$$

with

$$\begin{aligned} q_+^\beta(z) &\equiv \exp(-\mathrm{i}\omega_\nu^2 z)\,, && \beta \le \infty \\ q_-^\beta(z) &\equiv \begin{cases} 0\,, & \beta = \infty \\ \exp(-\beta\omega_\nu^2) \exp(-\mathrm{i}\omega_\nu^2 z)\,, & \beta < \infty\,. \end{cases} \end{aligned} \tag{A8}$$

In analogy with the discussion beginning with (2.45), we see that G_+ is holomorphic in the region $s<0$, and G_- in the region $s<\beta$. These functions are *not* two pieces of *one* holomorphic function, since the commutator (A6) does not vanish on an open interval of the t axis in the nonrelativistic case. Of course, the KMS condition (3.9) holds true again: G_- in the strip $0 \le s \le \beta$ is a copy of G_+ in the strip $-\beta \le s \le 0$. If we identify (say) $t - \mathrm{i}\beta/2$ with $t + \mathrm{i}\beta/2$, we obtain a cylindrical complex manifold on which is defined a function $\mathcal{G}^\beta(z, x, y)$ which is holomorphic everywhere except on the t axis, where it has two different boundary values. Similarly, for $\beta = \infty$ we put G_+^∞ at $s<0$ together with G_-^∞ at $s>0$ (namely, zero!) to get a $\mathcal{G}^\infty$ holomorphic except on $s=0$.

As in section 2, let

$$G^\beta(s, x, y) \equiv \mathcal{G}^\beta(\mathrm{i}s, x, y)\,. \tag{A9}$$

(It is not hard to see that G^β, viewed as a distribution in (s, x, y), does not depend on which boundary value of $\mathcal{G}^\beta$ at $s=0$ is taken. Thus we need not and shall not commit ourselves to a choice.) Let us first consider the case $\beta=\infty$. Then it is clear that $G^\infty(s_2 - s_1, x, y)$ solves

$$[-\partial/\partial s_2 + K_{(x)}]\, G = \delta(s_2 - s_1)\, \delta(x-y)\, \gamma(y)^{-1/2}\,. \tag{A10}$$

In fact, G^∞ is just the usual Green function for the (time-reversed) heat equation associated with $K_{(x)}$. We may also write

$$G^\infty(s, x, y) = (2\pi)^{-1} \sum_{\nu=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k_0\, \psi_\nu(x) \exp(\mathrm{i}k_0 s_2)\, \psi_\nu(y)^* \exp(-\mathrm{i}k_0 s_1)\, (-\mathrm{i}k_0 + \omega_\nu^2)^{-1}\,,$$
$$s \equiv s_2 - s_1 \neq 0\,. \tag{A11}$$

Thus, G^∞ can also be regarded as the kernel of the inverse to

$$-\partial/\partial s + K_{(x)} \tag{A12}$$

viewed as an operator on the Hilbert space $\mathrm{L}^2(\mathbf{R} \times \mathrm{M}; \gamma^{1/2}\, \mathrm{d}s\, \mathrm{d}^n x)$. This parallels the situation in the relativistic case; cf. section 2.2, esp. (2.25), (2.26).

We shall now show that, with our definition of G^β, the state of affairs for finite temperature is the same as in the relativistic case, too. Specifically, one has

$$G^\beta(s, x, y) = \beta^{-1} \sum_{\nu=1}^{\infty} \sum_{n=-\infty}^{\infty} \psi_\nu(x) \exp(2\pi \mathrm{i} n s_2/\beta)\, \psi_\nu(y)^* \exp(-2\pi \mathrm{i} n s_1/\beta) \left[-\frac{2\pi \mathrm{i} n}{\beta} + \omega_\nu^2\right]^{-1}, \tag{A13}$$

which is an immediate consequence of the readily verified equality

$$\sum_{n=-\infty}^{\infty} \left[-\frac{2\pi \mathrm{i} n}{\beta} + \omega^2\right]^{-1} \exp(2\pi \mathrm{i} n s/\beta) = \beta(1 - \exp(-\beta\omega^2))^{-1} \exp(-\beta\omega^2) \exp(\omega^2 s)\,, \quad 0 < s < \beta\,. \tag{A14}$$

Thus, G^β solves (A10), viewed as an equation on $S^1 \times M$ (where S^1 has circumference β), since it equals the kernel of the inverse of (A12), regarded as operating on functions in $L^2(S^1 \times M; \gamma^{1/2}\, ds\, d^n x)$. This is the analogue of the situation described in section 2.3 [see the paragraph containing (2.50b)].

If we take boundary values of $\mathcal{G}^\beta$ on the real z axis in the same way as described below (2.52), then we obtain the usually considered *time-ordered* Green function

$$G_F^\beta(t, x, y) \equiv \langle \mathcal{T}[\psi(t, x)\, \psi^\dagger(0, y)] \rangle_\beta = G_-^\beta(t, x, y) + \theta(t)\, [\psi(t, x), \psi^\dagger(0, y)]\,. \tag{A15}$$

It satisfies the inhomogeneous Schrödinger equation

$$(\mathrm{i}\partial/\partial t - K_{(x)})G = \mathrm{i}\delta(t)\, \delta(x - y)\, \gamma(y)^{-1/2}\,. \tag{A16}$$

These equations are the nonrelativistic counterparts to (2.53) and (2.54). Also, G_F^β again connects the boundary values of $\mathcal{G}^\beta$ from adjacent holomorphy strips. We repeat, however, that in the nonrelativistic case these boundary values are everywhere different, in sharp contrast to the relativistic case.

Finally, we note that the finite-temperature functions may again be written as image sums of the $T = 0$ functions, as in (2.58) and (2.59). This follows from the same arguments as those presented in section 2.5.

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