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Dynamics of Extended Bodies in General Relativity. III. Equations of Motion

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DYNAMICS OF EXTENDED BODIES IN GENERAL RELATIVITY

III. EQUATIONS OF MOTION†

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A study is made of the motion of an extended body in arbitrary gravitational and electromagnetic fields. In a previous paper it was shown how to construct a set of reduced multipole moments of the charge-current vector for such a body. This is now extended to a corresponding treatment of the energy-momentum tensor. It is shown that, taken together, these two sets of moments have the following three properties. First, they provide a full description of the body, in that they determine completely

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the energy-momentum tensor and charge-current vector from which they are constructed. Secondly, they include the total charge, total momentum vector and total angular momentum (spin) tensor of the body. Thirdly, the only restrictions on the moments, apart from certain symmetry and orthogonality conditions, are the equations of motion for the total momentum and spin, and the conservation of total charge. The time dependence of the higher moments is arbitrary, since the process of reduction used to construct the moments has eliminated those contributions to these moments whose behaviour is determinate. The uniqueness of the chosen set of moments is investigated, leading to the discovery of a set of properties which is sufficient to characterize them uniquely.

The equations of motion are first obtained in an exact form. Under certain conditions, the contributions from the moments of sufficiently high order are seen to be negligible. It is then convenient to make the *multipole approximation*, in which these high order terms are omitted. When this is done, further simplifications can be made to the equations of motion. It is shown that they take an especially simple form if use is made of the extension operator of Veblen & Thomas. This is closely related to repeated covariant differentiation, but is more useful than that for present purposes. By its use, an explicit form is given for the equations of motion to any desired multipole order. It is shown that they agree with the corresponding Newtonian equations in the appropriate limit.

1. Introduction

In the general theory of relativity, all material systems possess a symmetric energy-momentum tensor $T^{\alpha\beta}$ which occurs as the source term of the gravitational field equations

$$R^{\alpha\beta}-\tfrac{1}{2}Rg^{\alpha\beta}=\kappa T^{\alpha\beta}. \tag{1.1}$$

By the contracted Bianchi identity, these are consistent only if

$$\nabla_\beta T^{\alpha\beta}=0. \tag{1.2}$$

The four restrictions that this places on the ten components of $T^{\alpha\beta}$ have, as their nearest Newtonian equivalent, the equations of motion of a general continuum and its equation of energy conservation. But to make the motion determinate, one must add further restrictions which will be specific to the particular type of matter under consideration. This is usually achieved by expressing $T^{\alpha\beta}$ in terms of variables more characteristic of the particular system being studied, and then imposing equations of state on these new variables. For example, a simple perfect fluid may be described by its density ρ, pressure p and velocity u^α, with

$$T^{\alpha\beta}=\rho u^\alpha u^\beta+p(u^\alpha u^\beta-g^{\alpha\beta}). \tag{1.3}$$

Since $u^\alpha u_\alpha=1$, we have reduced the number of variables from ten to five. The system thus becomes determinate if we add to (1.2) an equation of state relating ρ to p.

If sufficient is known about the matter being studied, then it is always possible to proceed in this way. Suppose, however, that one is interested in the motion of a planet in the Sun's gravitational field. This is largely independent of the detailed internal structure of the planet, and to high accuracy can be determined in Newtonian theory from a few parameters, especially its total mass and principal moments of inertia. Such a problem needs a different approach. Many authors have attempted to develop equations of motion for extended bodies in general relativity that correspond to these Newtonian results. This Introduction discusses the difficulties that face any such attempt, and shows why previous treatments, all of which make a direct attack on the equations of motion, are unsatisfactory. In the remainder of the paper it will be shown how these difficulties can be overcome by approaching the problem from a different direction. Standard notations and conventions that will be used throughout the paper are summarized in appendix 1.

To form a guide to the features that we require of the relativistic theory, we first consider the corresponding Newtonian equations. For these, all tensor indices will be written as subscripts, latin indices which run from 1 to 3 being used. Let $z_a(t)$ be the Cartesian coordinates of a moving point about which all moments will be taken, and let $v_a = \mathrm{d}z_a/\mathrm{d}t$ be its velocity. This point will normally be chosen as the centre of mass of the body, but it is convenient for the time being to leave it arbitrary. If ρ is the mass density and u_a the material velocity at a general point of the body, then the total momentum p_a and angular momentum $S_{ab} = S_{[ab]}$ of the body are defined by

$$p_a = \int \rho u_a \, \mathrm{d}^3x, \quad S_{ab} = 2\int \rho r_{[a} u_{b]} \, \mathrm{d}^3x, \tag{1.4}$$

where $r_a = x_a - z_a$. The resulting equations of motion of the body, in a gravitational field of potential ϕ, are

$$\mathrm{d}p_a/\mathrm{d}t = -\int \rho \phi_a \, \mathrm{d}^3x \tag{1.5}$$

and

$$\mathrm{d}S_{ab}/\mathrm{d}t = 2p_{[a}v_{b]} - 2\int \rho r_{[a}\phi_{b]} \, \mathrm{d}^3x, \tag{1.6}$$

where $\phi_a = \partial_a \phi$. If the gravitational field varies only slowly through the region occupied by the body, ϕ can be approximated by the first few terms of its Taylor series about z_a. In this way we get the multipole expansions

$$\mathrm{d}p_a/\mathrm{d}t = -\sum \frac{1}{n!} m_{b_1 \ldots b_n} \partial_{ab_1 \ldots b_n} \phi \tag{1.7}$$

and

$$\mathrm{d}S_{ab}/\mathrm{d}t = 2p_{[a}v_{b]} - 2\sum \frac{1}{n!} m_{c_1 \ldots c_n [a} \partial_{b] c_1 \ldots c_n} \phi, \tag{1.8}$$

where

$$m_{a_1 \ldots a_n} = \int \rho r_{a_1} \ldots r_{a_n} \, \mathrm{d}^3x \quad \text{for} \quad n \geqslant 0. \tag{1.9}$$

The summations start at $n = 0$ and terminate at a value that gives a sufficiently good approximation to ϕ.

In addition to (1.7) and (1.8), the variables appearing in these equations must also satisfy

$$\mathrm{d}m/\mathrm{d}t = 0 \tag{1.10}$$

and

$$\mathrm{d}m_a/\mathrm{d}t = p_a - mv_a \tag{1.11}$$

in consequence of the mass conservation equation

$$\partial\rho/\partial t + \partial_a(\rho u_a) = 0. \tag{1.12}$$

This leaves the time dependence of v_a, and of $m_{a_1 \ldots a_n}$ for $n \geqslant 2$, undetermined. Since the base point $z_a(t)$ was arbitrarily chosen, we expect v_a to be arbitrary. If we choose it to be the centre of mass of the body by imposing $m_a = 0$, then (1.11) gives $p_a = mv_a$. The situation is different for the moments $m_{a_1 \ldots a_n}(t)$, $n \geqslant 2$. Their time dependence is determined by the internal structure of the body, and its motion is not determinate until further information is given about them. They are thus similar to the density ρ in (1.3), which must be specified as a function of the pressure before the motion of the fluid becomes determinate. They are also similar to ρ in their arbitrariness. The function $\rho(p)$ is specifiable arbitrarily, subject only to certain general inequalities such as the velocity of sound being less than that of light. It may not be possible to find a physical material with the prescribed behaviour, but that is a different problem. Similarly, the functions

$m_{a_1\ldots a_n}(t)$ are arbitrary for $n \geqslant 2$, subject only to certain general conditions arising from requiring $\rho \geqslant 0$ in (1.9).

It is this separation of the parameters determining the motion into determinate parameters m, p_a and S_{ab} describing the overall motion of the body in space, and arbitrarily specifiable parameters $m_{a_1\ldots a_n}$ $(n \geqslant 2)$ characterizing such behaviour as internally produced deformations of the body, that is the main feature of the Newtonian situation that we wish to reproduce in the relativistic theory. In Newtonian theory, a rigid body with angular velocity $\Omega_{ab}(t)$ has

$$u_a(\boldsymbol{x}, t) = v_a(t) + \Omega_{ab}(t)\, r_b.$$

If z_a is chosen as the centre of mass, this implies that

$$S_{ab} = 2m_{c[a}\Omega_{b]c} \tag{1.13}$$

and

$$\mathrm{d}m_{a_1\ldots a_n}/\mathrm{d}t = nm_{c(a_1\ldots a_{n-1}}\Omega_{a_n)c} \tag{1.14}$$

which, together with the above equations, make the motion determinate. Provided that relativistic parameters can be found having a similar arbitrariness, a relativistic 'ideal rigid body' can be characterized by adopting corresponding equations as definitions. Note that it is irrelevant whether or not such an ideal body exists – its use is to serve as an approximately realizable idealization, the effects of departure from which are small and can be separately investigated.

The desired separation can be achieved easily in Newtonian theory because the gravitational force is determined solely by the mass density. In a relativistic theory, all forms of energy have an equivalent passive gravitational mass. In particular, this applies both to the kinetic energy of internal motions and to the elastic energy of internal stresses. Since these are described respectively by the T^{0a} and T^{ab} components of the energy-momentum tensor, we see that the relativistic analogue of the m's appearing on the right hand sides of (1.7) and (1.8) must be moments constructed out of the full tensor $T^{\alpha\beta}$. The general idea is to pick an arbitrary timelike world line l, and to define moments of $T^{\alpha\beta}$ as tensor fields along l. These will include the total momentum p^κ and spin $S^{\kappa\lambda}$ of the body, and the equations of motion sought will be for the absolute derivatives $\delta p^\kappa/\mathrm{d}s$ and $\delta S^{\kappa\lambda}/\mathrm{d}s$, where s is a parameter along l and $\delta/\mathrm{d}s = (\mathrm{d}x^\alpha/\mathrm{d}s)\,\nabla_\alpha$.

The problem of defining a covariant set of moments of $T^{\alpha\beta}$ is easily solved. There are many ways of doing this, the following simple method being just one possibility. If $z^\lambda(s)$ is a parametrization of l, put $v^\lambda = \mathrm{d}z^\lambda/\mathrm{d}s$. To define moments at $s = s_0$, set up a normal coordinate system with pole at $z_0 = z(s_0)$ and such that

$$v^\lambda(s_0) = \delta^\lambda_0, \quad g_{\lambda\mu} = \mathrm{diag}\,(1, -1, -1, -1) \quad \text{at} \quad z_0. \tag{1.15}$$

Put

$$t^{\kappa_1\ldots\kappa_n\lambda\mu}(s_0) = \int x^{\kappa_1}\ldots x^{\kappa_n}T^{\lambda\mu}(x)\,\surd(-g)\,w^\alpha\,\mathrm{d}\Sigma_\alpha, \tag{1.16}$$

where w^α is some canonically defined vector field and the integral is over the spacelike hypersurface $x^0 = 0$. The 2^n-pole moment of $T^{\alpha\beta}$ at $s = s_0$ is then taken to be the unique tensor at z_0 which reduces to this value in any such normal coordinate system. It is orthogonal to v^κ on its first n indices, and so has $5(n+1)\,(n+2)$ linearly independent components. These moments are generally considered in relation to a second set, with one fewer index, defined similarly by requiring

$$p^{\kappa_1\ldots\kappa_n\lambda}(s_0) = \int x^{\kappa_1}\ldots x^{\kappa_n}T^{\lambda\alpha}(x)\,\surd(-g)\,\mathrm{d}\Sigma_\alpha \tag{1.17}$$

to hold in the same normal coordinate system. These have the same orthogonality property, with $2(n+1)\,(n+2)$ linearly independent components. The momentum vector may be defined by

(1.17) with $n = 0$, while the spin tensor may be taken as $S^{\kappa\lambda} = 2p^{[\kappa\lambda]}$. Equation (1.2) can then be used to evaluate $\delta p^{\kappa}/\mathrm{d}s$ and $\delta S^{\kappa\lambda}/\mathrm{d}s$ in terms of the t's.

If the t's could be prescribed arbitrarily, the problem would then be solved. Unfortunately, this is not so since there are an infinite number of relations between them, also as a consequence of (1.2). It is thus necessary to re-express the equations of motion in terms of an alternative set of moments which is free from such interrelations. We shall call such a set the *reduced moments* of $T^{\alpha\beta}$, in contrast to such a set as the t's which we call *complete moments*. Since the whole of $T^{\alpha\beta}$ is involved in determining gravitational effects on the motion, these reduced moments must, like the t's, actually contain enough information to completely determine the $T^{\alpha\beta}$ from which they are constructed.

A simple illustration of a reduced moment is provided by considering the Newtonian dipole moment of the momentum density ρu_a. The complete dipole moment p_{ab} is defined, in the notation of (1.4), by

$$p_{ab} = \int \rho r_a u_b \, \mathrm{d}^3x. \tag{1.18}$$

However, it follows from (1.12) that

$$p_{(ab)} = m_{(a} v_{b)} + \tfrac{1}{2}\mathrm{d}m_{ab}/\mathrm{d}t. \tag{1.19}$$

Hence the m's, which we assume known, determine $p_{(ab)}$, and it is thus only necessary to give separately its antisymmetric part. This is precisely the spin tensor, $S_{ab} = 2p_{[ab]}$, so that the use of S_{ab} instead of p_{ab} is an example of the reduction process.

The equations corresponding to (1.19) in the relativistic theory, needed to start the reduction of the moments $t^{\cdots}$ of (1.16), each contain an infinite number of terms. This makes any direct approach to the reduction process extremely difficult. It seems that to make any progress at all, it is necessary to neglect all moments higher than a certain low order, e.g. dipole or quadrupole order. Even then, the quadrupole case still presents difficulties, as will be seen below. Previous attacks on the problem, such as those of Mathisson (1937), Papapetrou (1951), Tulczyjew (1959), Dixon (1964), Taub (1964) and Madore (1969), have all been based on such a cutoff. This may be formalized as a limit in which the size of the body is shrunk to zero. However, the intended applications are as approximations to the motion of a body of small but nonzero extent. In this case the neglect of the higher moments must be based on the smoothness of the field, as in the derivation of (1.7) and (1.8). This in turn relies on the order of the derivative of the field that occurs in combination with a particular moment in the equations of motion increasing as the order of the moment increases, as is also the case in (1.7) and (1.8). Although it is intuitively reasonable that this should happen, it actually does so only if the definitions of the moments are suitably chosen.

These points will be illustrated from the results of Taub (1964) and Madore (1969), who both use a cutoff at the quadrupole terms. Similar difficulties are present in the methods used in the other papers mentioned above, but they all make a cutoff at the dipole terms, and it is only at the quadrupole level that the main problems of the method become apparent. An additional complication is that both Taub and Madore have non-trivial algebraic errors in their calculations which affect their final equations of motion. The results that we quote below use the corrected versions. We shall write p^{κ} and $S^{\kappa\lambda}$ for the momentum and spin tensors used by the author quoted, without entering into their precise definition. In both cases it is along the lines of the above discussion, and the precise form is immaterial for our purposes.

We begin with the results of Madore. He uses a three-index quadrupole tensor $M^{\kappa\lambda\mu} = M^{(\kappa\lambda)\mu}$ and obtains

$$\delta p_\kappa/\mathrm{d}s = R_{\kappa\lambda\mu\nu}(\tfrac{1}{2}v^\lambda S^{\mu\nu} + \tfrac{1}{3}\delta M^{\lambda\mu\nu}/\mathrm{d}s) - \tfrac{2}{3}v^{(\lambda}M^{\rho)\nu\mu}\nabla_\rho R_{\kappa(\lambda\mu)\nu} \tag{1.20}$$

and

$$\delta S^{\kappa\lambda}/\mathrm{d}s = 2p^{[\kappa}v^{\lambda]} + \tfrac{4}{3}R^{[\kappa}{}_{\cdot(\mu\nu)\rho}M^{\lambda]\rho\mu}v^\nu, \tag{1.21}$$

where $v^\lambda = \mathrm{d}z^\lambda/\mathrm{d}s$. Madore's error, originating in his equation (41), has the effect of omitting the symmetrization over λ and μ present in the final term on the right hand side of (1.20). We see that (1.20) contains a term in which the quadrupole moment interacts with an undifferentiated curvature tensor. This is the expected behaviour for a dipole, rather than a quadrupole, term. It gives the first indication that the method violates the requirement that higher order moments interact with higher order derivatives of the field. The origin of this unwanted term lies with a poor choice of definition for the total momentum p_κ. If the equations are rewritten in terms of $^*p_\kappa$ instead, where

$$^*p_\kappa = p_\kappa - \tfrac{1}{3}M^{\lambda\mu\nu}R_{\kappa\lambda\mu\nu}, \tag{1.22}$$

then $M^{\lambda\mu\nu}$ does appear only in combination with the derivative $\nabla_\rho R_{\kappa\lambda\mu\nu}$ as required. Since the validity of using a multipole cutoff for bodies small in comparison with the length scale of the external field depends on this requirement, this illustrates clearly the need for care in the definition of the moments.

A difficulty of a different nature arises over the definition of the quadrupole moment. $M^{\lambda\mu\nu}$ is essentially constructed from the $p^{\lambda\mu\nu}$ of (1.17). This involves only the energy and momentum components of $T^{\alpha\beta}$, with no contribution from the internal stresses. It thus cannot contain enough information for the complete quadrupole moment $t^{\lambda\mu\nu\rho}$ to be reconstructed, in the way that (1.19) enables p_{ab} to be reconstructed from S_{ab}. This shows up in Madore's derivations. He has to assume that certain contributions to the equations of motion, arising from the gravitational interaction with the internal stresses, are comparable with the neglected octopole terms although they formally appear to be of quadrupole order. They are then also neglected. This is accomplished by adopting a formula expressing $t^{\lambda\mu\nu\rho}$ in terms of $M^{\lambda\mu\nu}$ that would be valid if the stress terms were negligible. Although Madore claims that this is necessary only in the derivation of (1.21), correction of the error mentioned above in the derivation of (1.20) necessitates using it also for that equation. Although not stated by Madore, this adopted formula actually places a further restriction on $M^{\lambda\mu\nu}$, implying that it must have the form

$$M^{\lambda\mu\nu} = 2I^{\lambda\mu}v^\nu - 4I^{\nu(\lambda}v^{\mu)}, \tag{1.23}$$

where

$$I^{\lambda\mu} = I^{\mu\lambda} \quad \text{and} \quad p_\mu I^{\lambda\mu} = 0. \tag{1.24}$$

The quadrupole moment is thus being described by a six component tensor $I^{\lambda\mu}$ analogous to the Newtonian m_{ab} of (1.9), which also has six components. Thus the momentum contributions to the quadrupole moment, from T^{0a}, are also neglected in this treatment.

Taub similarly introduces an assumption concerning the quadrupole moment which is additional to those due to the quadrupole cutoff. In contrast to Madore, he explicitly uses it to reduce the corresponding tensor to a symmetric $I^{\lambda\mu}$, but one satisfying $v_\mu I^{\lambda\mu} = 0$ instead of $p_\mu I^{\lambda\mu} = 0$. If $S^{\kappa\lambda}$ is identified with Taub's $2L^{[\kappa\lambda]}$, his equations of motion are

$$\delta p_\kappa/\mathrm{d}s = R_{\kappa\lambda\mu\nu}(\tfrac{1}{2}v^\lambda S^{\mu\nu} + \delta\,(I^{\lambda\mu}v^\nu)/\mathrm{d}s) - v^\lambda v^\mu I^{\nu\rho}\nabla_\rho R_{\kappa\lambda\mu\nu} \tag{1.25}$$

and

$$\delta S^{\kappa\lambda}/\mathrm{d}s = 2p^{[\kappa}v^{\lambda]} - R^{[\kappa}{}_{\cdot\mu\nu\rho}(4I^{\lambda]\nu}v^\mu v^\rho - 2v^{\lambda]}v^\rho I^{\mu\nu}). \tag{1.26}$$

His error appears first in his equation (2.9), and has the effect of replacing the coefficients 4 and 2 in the final bracket of (1.26) by 2 and 0 respectively. The momentum equation again contains a term in which the quadrupole moment interacts with an undifferentiated curvature tensor, and again it is removable by using a different momentum vector $*p_\kappa$. This time we need to put

$$*p_\kappa = p_\kappa - R_{\kappa\lambda\mu\nu} I^{\lambda\mu} v^\nu. \tag{1.27}$$

Even if the neglect of the momentum and stress contributions to the quadrupole interaction can be justified when a quadrupole cutoff is used, it is only pushing away by one step the problems that they introduce. These will certainly have to be faced if the octopole case is to be treated. What is essentially happening is that a guess is being made, namely that the reduced quadrupole moment tensor is essentially the $p^{\kappa\lambda\mu}$ of (1.17), and it is turning out to be incorrect. To make it work, it is then necessary to suppose that the quadrupole structure of the body is of a specially simple kind, describable by the six-component tensor $I^{\lambda\mu}$.

The balance between arbitrarily specifiable moments, and moments adequate to determine the motion, is a very fine one. Since the reduced moments must strike this balance, they are unlikely to be found by *a priori* guesses. The above discussion shows that the quadrupole case is about the limit at which anything useful can be achieved in this way, and even the general quadrupole seems beyond its reach. To make further progress the problem must be turned round, with attention directed primarily on finding an appropriate set of reduced moments for $T^{\alpha\beta}$. When this has been done, the equations of motion should follow naturally. The present paper adopts this approach, and shows that it avoids the need to make any cutoff at all in the multipole series. We shall determine the reduced moments of all orders, and shall explicitly find equations of motion in terms of them, correct to any desired order. For comparison with the results discussed above, the equations we obtain for a general quadrupole are

$$\delta p_\kappa/\mathrm{d}s = \tfrac{1}{2} v^\lambda S^{\mu\nu} R_{\kappa\lambda\mu\nu} + \tfrac{1}{6} J^{\lambda\mu\nu\rho} \nabla_\kappa R_{\lambda\mu\nu\rho} \tag{1.28}$$

and

$$\delta S^{\kappa\lambda}/\mathrm{d}s = 2p^{[\kappa}v^{\lambda]} - \tfrac{4}{3} R^{[\kappa}{}_{.\mu\nu\rho} J^{\lambda]\mu\nu\rho}. \tag{1.29}$$

The quadrupole moment tensor $J^{\lambda\mu\nu\rho}$ has the symmetries of the curvature tensor, and thus has 20 linearly independent components. This is less than the 60 of the complete moment $t^{\lambda\mu\nu\rho}$ of (1.16), as is expected due to the reduction process. It should be compared with the 24 of Madore's initial choice $M^{\lambda\mu\nu}$, and the 6 that are left after $M^{\lambda\mu\nu}$ has been forced into a role for which it is inadequate. Both Madore's and Taub's results are special cases of (1.28) and (1.29), obtained by taking p_κ to be the $*p_\kappa$ of (1.22) or (1.27) as appropriate, and by setting

$$J^{\lambda\mu\nu\rho} = -6 v^{[\lambda} I^{\mu][\nu} v^{\rho]} \tag{1.30}$$

in both cases. Note that there is a qualitative difference between the effect of the reduction process on the monopole and dipole moments, and on the quadrupole moment. In the first two cases the number of indices is decreased by the reduction, $t^{\lambda\mu}$ and $t^{\lambda\mu\nu}$ yielding p^λ and $S^{\lambda\mu}$ respectively. For the quadrupole, however, the effect is to increase the symmetry of the tensor without changing the number of indices. This will be seen to hold also for all higher moments, the resulting reduced 2^n-pole moment ($n \geqslant 2$) having $(n+2)(3n-1)$ linearly independent components in comparison with the $(n+2)(5n+5)$ of the corresponding complete moment.

Let us now consider how the problem of finding the reduced moments of $T^{\alpha\beta}$ might be approached. We note first that it is not necessary to have gravitational forces present to make this a non-trivial problem, but in their absence the emphasis changes as these moments will no longer affect the motion of the body. The requirement that they must be sufficient to determine the motion must be replaced by one requiring them to be sufficient to completely determine the $T^{\alpha\beta}$ from which they are constructed. A guide to the gravitational case may thus be found by first considering the special relativistic theory of a body moving under electromagnetic forces. This case has been previously treated by the present author (Dixon 1967). Unfortunately, while the results obtained will be useful for this purpose, the methods used there are not extendable to deal also with the gravitational case.

The present paper is the third in a series devoted to tackling the additional problems which gravitation introduces. The two preceding papers in the series (Dixon 1970*a, b*) will be referred to as I and II respectively. The first of these studied in isolation the problem of defining the total momentum and spin of the body. This used as a guide the requirement that to every symmetry of the external fields there should correspond a conserved linear function of these variables. It was shown that this leads to essentially unique definitions, whose consequences were then studied in more detail. The second paper treated, in curved spacetime, the question of defining reduced moments for the charge-current vector J^α of an electrically charged extended body. Since this satisfies

$$\nabla_\alpha J^\alpha = 0, \tag{1.31}$$

it is analogous to, but simpler than, the corresponding problem for a $T^{\alpha\beta}$ satisfying (1.2). The analogue of the equations of motion for p^κ and $S^{\kappa\lambda}$ is, for J^α, the equation of conservation of total charge q. This is simply $\mathrm{d}q/\mathrm{d}s = 0$, which does not involve the higher moments of J^α.

The treatment of $T^{\alpha\beta}$ in the present paper draws upon II for guidance in dealing with a curved spacetime, and upon Dixon (1967) for guidance as to the properties to be expected of the reduced moments of $T^{\alpha\beta}$. It links with I through the appearance of the p^κ and $S^{\kappa\lambda}$ of I as a natural consequence of the reduction process, thus further supporting the definitions proposed there. It is convenient to treat the more general case of a body moving under the influence of both gravitational and electromagnetic forces, as the additional complications introduced by the electromagnetic field are small in comparison with those already present in the gravitational case, and it enables a ready comparison to be made of gravitational and electromagnetic effects. We thus replace (1.2) by

$$\nabla_\beta T^{\alpha\beta} = -F^{\alpha\beta} J_\beta, \tag{1.32}$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. A further generalization is required of the procedure used in II for J^α. To motivate it, an outline is given of the theory for J^α, presented in the form closest to that to be used for $T^{\alpha\beta}$. Where steps in the treatment of $T^{\alpha\beta}$ follow the corresponding steps for J^α without any novel features, they will be omitted and reference should be made to II for the method of proof.

The equations of motion that are finally obtained are

$$\begin{aligned}\delta p_\kappa/\mathrm{d}s = {} & \tfrac{1}{2}v^\lambda S^{\mu\nu}R_{\kappa\lambda\mu\nu} - qv^\lambda F_{\kappa\lambda} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}I^{\nu_1\ldots\nu_n\lambda\mu}\nabla_\kappa g_{\lambda\mu,\nu_1\ldots\nu_n} \\ & + \sum_{n=1}^{\infty}\frac{n}{(n+1)!}m^{\nu_1\ldots\nu_n\lambda}\nabla_\kappa F_{\lambda(\nu_1,\nu_2\ldots\nu_n)}\end{aligned} \tag{1.33}$$

and

$$\begin{aligned}\delta S^{\kappa\lambda}/\mathrm{d}s = {} & 2p^{[\kappa}v^{\lambda]} + \sum_{n=1}^{\infty}\frac{1}{n!}g^{\sigma[\kappa}I^{\lambda]\rho_1\ldots\rho_n\mu\nu}g_{\{\sigma\nu,\mu\}\rho_1\ldots\rho_n} \\ & + 2\sum_{n=0}^{\infty}\frac{1}{n!}g^{\sigma[\kappa}m^{\lambda]\rho_1\ldots\rho_n\mu}F_{\sigma\mu,\rho_1\ldots\rho_n}.\end{aligned} \tag{1.34}$$

The sums to infinity are formal, in that in general they will not converge. They should be cut off at an appropriate order, just as for the Newtonian equations (1.7) and (1.8). We shall, however, also obtain an exact version of these equations in which no such convergence problem arises. $I^{\lambda_1\ldots\lambda_n\mu\nu}$ is, for $n \geqslant 2$, the 2^n-pole moment tensor of $T^{\alpha\beta}$, while $m^{\lambda_1\ldots\lambda_n\mu}$ for $n \geqslant 1$ is the corresponding tensor for J^α. Indices following a comma denote the tensor extension operation of Veblen & Thomas (1923). This is related to repeated covariant differentiation, but is more convenient than that for our purposes. It is defined in appendix 2, and techniques are given there for evaluating such extensions in terms of more commonly used quantities. The symmetrizing bracket notations used above are defined in appendix 1. The moments satisfy

$$\left.\begin{aligned} I^{\lambda_1\ldots\lambda_n\mu\nu} = I^{(\lambda_1\ldots\lambda_n)(\mu\nu)} \quad \text{and} \quad I^{(\lambda_1\ldots\lambda_n\mu)\nu} = 0 \quad \text{for} \quad n \geqslant 2,\\ n_{\lambda_1} I^{\lambda_1\ldots\lambda_{n-2}[\lambda_{n-1}[\lambda_n\mu]\nu]} = 0 \quad \text{for} \quad n \geqslant 3 \end{aligned}\right\} \tag{1.35}$$

and

$$\left.\begin{aligned} m^{\lambda_1\ldots\lambda_n\mu} = m^{(\lambda_1\ldots\lambda_n)\mu} \quad \text{and} \quad m^{(\lambda_1\ldots\lambda_n\mu)} = 0 \quad \text{for} \quad n \geqslant 1,\\ n_{\lambda_1} m^{\lambda_1\ldots\lambda_{n-1}[\lambda_n\mu]} = 0 \quad \text{for} \quad n \geqslant 2, \end{aligned}\right\} \tag{1.36}$$

and have, respectively, $(n+2)(3n-1)$ and $\frac{1}{2}(n+2)(3n+1)$ linearly independent components. The tensor $J^{\kappa\lambda\mu\nu}$ of (1.28) and (1.29) is related to these by

$$J^{\kappa\lambda\mu\nu} = I^{[\kappa[\mu\lambda]\nu]} \quad \text{and} \quad I^{\kappa\lambda\mu\nu} = \tfrac{4}{3} J^{(\kappa|\mu|\lambda)\nu}. \tag{1.37}$$

The vector field n_λ is timelike but otherwise arbitrary along l, forming together with l the reference system with respect to which the moments are defined. Both n_λ and l are simultaneously determined by a suitable definition of the centre of mass of the body, as discussed in I. The close correspondence between the electromagnetic and gravitational terms in (1.33) and (1.34) is evident.

During the detailed development of the theory, we shall need certain mathematical techniques involving the use of vector bundles. To avoid these interrupting the main body of the work, these are presented separately in the following two sections. In the development given in II, it also appeared natural to express certain constructions in the language of fibre bundles. This was almost entirely for descriptive purposes, however, and arose as certain naturally occurring functions were functions on the tangent bundle TM of the spacetime manifold M. This will again be so in the present paper, but in certain parts of the development it will be necessary to use results from the differential geometry of such bundles. This is to enable us to covariantly differentiate tensorial functions on TM with respect to their base point in M, for which we need a connexion on the appropriate bundle over TM. In particular, the exact form of the equations of motion involves this operation. Although it is possible to express the results without using such operators, they then seem artificial, and it is clear that bundle theory provides the natural language in which to express them. We thus give in § 2 a brief account of those aspects of bundle theory that we shall need. This is restricted to vector bundles, and for those properties of connexions that we shall use, we follow the connexion map formalism of Dombrowski (1962). Although less well known than the more general theory of Kobayashi (1957) using principal bundles, it is better suited for our purposes as the introduction of principal bundles would only be an unnecessary complication. Section 2 also serves to introduce the notation of bundle theory that will be used in the rest of the paper.

Section 3 develops the relation between two-point tensors on M and certain fields defined on the tangent bundle. The results obtained here, and in the latter part of § 2, are essential to the derivation of the equations of motion, but they do not find their main application until § 12, apart from in that small portion of § 8 which forms the basis of § 12.

The development of the theory of moments starts in §4, and is based on certain moment generating functions. In §§4 and 5 we introduce these functions and associate with them certain functionals on TM. At this stage no attempt is made to link the moments with the fields $T^{\alpha\beta}$ and J^{α} which they are eventually to describe. Instead, certain properties are assumed for the moments as working hypotheses, based on the known situation in special relativity, and their consequences are then investigated to suggest ways of linking the moments to their corresponding source fields. In §6 a number of preliminary results are obtained to assist in this linking process.

Up to this stage the theories of the moments of both J^{α} and $T^{\alpha\beta}$ are developed together. In §7 we separate the two by completing the development of J^{α}. The method starts by first hypothesizing a provisional linkage between J^{α} and its moments. From this, not only are the existence and uniqueness deduced of moments having the properties assumed in §4, but also a result is obtained which forms the basis of a new, and better, definition for the moments. The existence and uniqueness properties are then proved from the new definition by reducing them to those already proved for the provisional definition.

This roundabout method is not really necessary for dealing with J^{α}. Section 7 just rederives results obtained in II by directly assuming, and working from, the result eventually adopted in §7 as the final definition. But by taking this as a model, the more difficult case of $T^{\alpha\beta}$ can be treated, for which it does not seem possible to omit the stepping stone provided by the provisional linkage. The provisional linkage for $T^{\alpha\beta}$ is given and studied in §8. In the course of this, the equations of motion for the momentum and angular momentum are obtained, but further study of these is deferred until §12. The development from the provisional definition is completed in §9 by a derivation of explicit expressions for the moments as integrals over spacelike cross-sections of the body.

By this point we are in a position to give the final characterization of the reduced moments of $T^{\alpha\beta}$. Since this is one of the main results of the paper, it is given in §10 in the form of a fairly self-contained theorem. This is intended to act also as a summary of the results of the previous six sections. Section 11 then completes the proof of this theorem.

The equations of motion that were obtained in §8 are studied in more detail in §12. This treatment maintains their exact form, which is useful for further analytical development, but for practical use it is more convenient to have approximate, but more manageable, equations. Such equations are found in §13 by making the multipole approximation, thus obtaining (1.33) and (1.34). They are approximate only in so far as the infinite series involved do not in general converge. The contribution from each multipole order is exact. The paper concludes with a general discussion of the results in §14.

It is suggested that the reader turns next to §§10 and 13 for a more detailed statement of the main results of the paper. With §10 acting as an introduction to the notation, the development can then be picked up from §4 onwards. Sections 2 and 3 can be studied at any stage, but as mentioned above, the most powerful results of those sections are not needed until the equations of motion are studied in detail in §§12 and 13.

2. Results from the theory of vector bundles

We give now a brief account of those definitions and results from the theory of vector bundles that we shall need later. A more detailed account of vector bundle theory, avoiding the use of general fibre bundles, is given by Lang (1962). It does not include the theory of connexions, but an account of this in the spirit of Lang's book is given by Vilms (1967), using the connexion map approach of Dombrowski (1962). Throughout this section, all manifolds and mappings between manifolds will be assumed to be of class C^∞. We start with some definitions.

Let $GL(V)$ denote the group of all nonsingular linear transformations of an n-dimensional real vector space V. If E and M are manifolds, a map $\pi: E \to M$ is called an n-dimensional real *vector bundle* if it satisfies these two conditions:

(i) There exists an open covering $\{U_i\}$ of M and, for each i, a diffeomorphism

$$\phi_i: U_i \times V \to \pi^{-1}(U_i) \quad \text{such that} \quad \pi \circ \phi_i(x,u) = x \quad \text{for all} \quad (x,u) \in U_i \times V.$$

(ii) If $x \in U_i$, let $\phi_{ix}: V \to \pi^{-1}(x)$ be defined by $\phi_{ix}(u) = \phi_i(x,u)$. Then we require that

$$\phi_{jx}^{-1} \circ \phi_{ix} \in GL(V) \quad \text{for all} \quad x \in U_i \cap U_j,$$

and that the map $\psi_{ji}: U_i \cap U_j \to GL(V)$ defined by $\psi_{ji}(x) = \phi_{jx}^{-1} \circ \phi_{ix}$ be of class C^∞.

E is called the bundle space, M the base space and π the projection of the bundle. By abuse of language we sometimes describe E as being a bundle over M. If $x \in M$, $E_x := \pi^{-1}(x)$ is called the fibre over x. If $x \in U_i$, then ϕ_{ix} naturally induces a vector space structure on E_x, which by (ii) is independent of the choice of U_i.

A *cross-section* of the bundle π is a map $f: M \to E$ such that $\pi \circ f: M \to M$ is the identity map. Let $\gamma: I \to M$ be a curve in M, where I is an open interval of the real line R. A *lift* of γ to E is a curve $\beta: I \to E$ in E such that $\gamma = \pi \circ \beta$. The cross-sections of π clearly form an infinite dimensional vector space, with addition and scalar multiplication being performed pointwise in the vector space structure of the fibres. The same is true of the lifts of a given curve in M.

Let T_x be the tangent space to M at $x \in M$. Then $TM := \bigcup_{x \in M} T_x$ can be made into a vector bundle over M, called the tangent bundle of M. Define the projection $\tau: TM \to M$ by $\tau(X) = x$ if $X \in T_x$. Let $\{U_i, \theta_i\}$ be an atlas of local coordinates on M, so that $\{U_i\}$ is an open covering of M and $\theta_i: U_i \to R^m$ is one-to-one, where $m = \dim M$. Then $\{\tau^{-1}(U_i)\}$ is a covering of TM. We give TM the structure of a $2m$-dimensional manifold by specifying, as the local coordinates of $X \in \tau^{-1}(U_i)$, the m coordinates $\theta_i \circ \tau(X)$ of the base point together with the m components of X with respect to θ_i. With this structure, $\tau: TM \to M$ becomes an m-dimensional vector bundle. Its cross-sections are the vector fields on M, and a lift of a curve γ in M is simply a vector field along γ. In a similar manner we can also construct the tensor bundle $\tau^r_s: T^r_s(M) \to M$ of type (r,s). This is formed from all tensors on M of contravariant degree r and covariant degree s. Then $\tau^1_0 = \tau$, the tangent bundle, and τ^0_1 is called the cotangent bundle of M.

If V is a vector space, the tangent space $T_u(V)$ at each point $u \in V$ is naturally isomorphic to V. There is thus a canonical isomorphism $TV \to V \oplus V$ of the tangent bundle of V under which $X \mapsto (u,v)$ if $X \in T_u(V)$ and v is the image of X under the above natural isomorphism. Similarly the tensor bundle $T^r_s(V)$ is canonically isomorphic to $V \oplus V^r_s$, where V^r_s is the tensor space over V of type (r,s). The bundle structure is then trivial in both these cases. We shall need these observations below.

Suppose $\pi: E \to M$ and $\rho: F \to N$ are vector bundles, not necessarily of the same dimension. Given maps $\alpha: F \to E$ and $\beta: N \to M$, we say that α is a bundle map over β if $\pi \circ \alpha = \beta \circ \rho$ and if, for each $y \in N$, α maps F_y linearly into $E_{\beta(y)}$. Now suppose further that α induces an isomorphism of these fibres, which does require the bundles to have the same dimension. If $h: N \to F$ is a cross-section of ρ, put $\bar{h} := \alpha \circ h$, so that $\pi \circ \bar{h} = \beta$. Then $h \mapsto \bar{h}$ is a one-to-one correspondence between all cross-sections h of ρ and all maps $\bar{h}: N \to E$ satisfying $\pi \circ \bar{h} = \beta$. In particular, if f is a cross-section of π, then $\bar{h} := f \circ \beta$ satisfies this condition. The corresponding cross-section of ρ is denoted by $\alpha^{\#}f$, and is said to be induced from f by α.

Next we have a technique for constructing new bundles from a given one. Suppose we are given a vector bundle $\pi: E \to M$, a manifold N and a map $\beta: N \to M$. Let ρ and σ be the projections of the product manifold $N \times E$ onto N and E respectively, and let $N \times_M E$ denote the subset of $N \times E$ diagonal over M, i.e. $(y, u) \in N \times_M E$ if $\beta(y) = \pi(u)$. The bundle structure of π ensures that $N \times_M E$ is a submanifold of $N \times E$. Let $\beta^{\#}\pi$ and $\pi^{\#}\beta$ be the restrictions of ρ and σ respectively to $N \times_M E$. Then $\beta^{\#}\pi$: $N \times_M E \to N$ is a vector bundle, said to be induced from π by β, and $\pi^{\#}\beta: N \times_M E \to E$ is a bundle map over β which maps fibres isomorphically. From our above remarks, the cross-sections h of $\beta^{\#}\pi$ are thus in one-to-one correspondence with the maps $\bar{h}: N \to E$ satisfying $\pi \circ \bar{h} = \beta$, this correspondence now being given by

$$h \mapsto \bar{h} = (\pi^{\#}\beta) \circ h. \tag{2.1}$$

We shall frequently represent cross-sections of induced bundles in this way, as it avoids the use of the manifold $N \times_M E$, which is rather cumbersome in notation. If $E = T^r_s(M)$ and $\pi = \tau^r_s$, we shall follow Bishop & Goldberg (1968, §5.9) in describing such a map $\bar{h}$ as a tensor field of type (r, s) *on* N *over* β.

We now introduce some notation concerning derivatives of maps. If $\gamma: I \to N$ is a curve in a manifold N, and $t \in I$, the tangent vector to γ at t is denoted by $\gamma'(t)$. Hence $\gamma': I \to TN$ is a lift of γ to TN, called the canonical lift. If $\phi: N \to M$ is a map of N into another manifold M, the derivative ϕ_* of ϕ is the unique map $\phi_*: TN \to TM$ such that $\phi_* \circ \gamma' = (\phi \circ \gamma)'$ for all curves γ in N. It is a bundle map over ϕ of the corresponding tangent bundles. In terms of local coordinates $\{y^\alpha\}$ and $\{x^k\}$ on N and M respectively, if $Y \in T_y(N)$ and $X = \phi_* Y \in T_x(M)$ with $x = \phi(y)$, then

$$X^k = (\partial x^k / \partial y^\alpha)\, Y^\alpha.$$

If ϕ is a diffeomorphism, then ϕ_* is a vector space isomorphism on corresponding tangent spaces. But an isomorphism between two vector spaces can be uniquely extended to an algebra isomorphism of the corresponding tensor algebras. By so extending ϕ_* at each point of N, we obtain a bundle map $\phi_A: T^r_s(N) \to T^r_s(M)$ over ϕ of the tensor bundles of arbitrary type. The map ϕ_A will also be a diffeomorphism, and we shall denote its inverse by ϕ^A. Acting on the cotangent bundles, ϕ^A is just the transpose of ϕ_*. Using the same local coordinates as before, if $Q \in (T^r_s)_y(N)$ and $P = \phi_A Q \in (T^r_s)_x(M)$, then $Q = \phi^A P$ and

$$P^{k\cdots}{}_{l\cdots} = \frac{\partial x^k}{\partial y^\alpha} \cdots \frac{\partial y^\beta}{\partial x^l} \cdots Q^{\alpha\cdots}{}_{\beta\cdots}, \quad Q^{\alpha\cdots}{}_{\beta\cdots} = \frac{\partial y^\alpha}{\partial x^k} \cdots \frac{\partial x^l}{\partial y^\beta} \cdots P^{k\cdots}{}_{l\cdots}. \tag{2.2}$$

If M is an m-dimensional manifold and $r \leqslant m$, an r-dimensional (differentiable) *distribution* S on M is an assignment to each $x \in M$ of an r-dimensional subspace S_x of $T_x(M)$, in such a way that S is locally spanned by r differentiable vector fields.

Now let $\pi: E \to M$ be an n-dimensional vector bundle over M, so that $\dim E = m + n$, and let τ_M: $TM \to M$ be the tangent bundle of M. Then TE has two natural bundle structures, for π_*: $TE \to TM$ gives it the structure of a $2n$-dimensional vector bundle, in addition to its $(m+n)$-dimensional bundle structure as the tangent bundle τ_E: $TE \to E$ of the manifold E. To define the vector space structure on the fibres of π_*, let $X \in TM$ and U, $V \in \pi_*^{-1}(X)$. Suppose γ is a curve in M such that $X = \gamma'(0)$, and let Γ_1 and Γ_2 be lifts of γ to E satisfying $U = \Gamma'_1(0)$, $V = \Gamma'_2(0)$.

Then if $a, b \in R$, we define $aU+bV = (a\Gamma_1+b\Gamma_2)'(0)$. The maps π_* and τ_E thus both play a dual role. For π_* is both a bundle with base TM, and a bundle map of τ_E to τ_M over π, while similarly τ_E is both a bundle with base E, and a bundle map of π_* to π over τ_M.

The set of vectors $V \in TE$ satisfying $\pi_* V = 0$ forms an n-dimensional distribution $\mathscr{V}$ on E, called the *vertical distribution*. If $x \in M$ and $w \in E_x$, then $\mathscr{V}_w = T_w(E_x)$, which is a subspace of $T_w(E)$. An m-dimensional distribution $\mathscr{H}$ on E is said to be *horizontal* if $\mathscr{V}_w \cap \mathscr{H}_w = \{0\}$ for all $w \in E$, in which case π_* maps $\mathscr{H}_w$ isomorphically onto $T_x := T_x(M)$. Since then $T_w(E) = \mathscr{H}_w \oplus \mathscr{V}_w$, any $U \in T_w(E)$ can be uniquely expressed as the sum of a horizontal and a vertical component, and the horizontal component is uniquely determined by $\pi_* U \in T_x$. We now define a bundle map $D: TE \to E$ of τ_E to π over π such that DU determines the vertical component of U. D is the composition of two maps. We first map U into its vertical component, which lies in $T_w(E_x)$, and then map this into E_x by the canonical isomorphism of these two spaces. The map $U \mapsto (\pi_* U, DU)$ is then an isomorphism of $T_w(E)$ onto $T_x \oplus E_x$.

Although by construction D is a bundle map of τ_E to π over π, in general it will not be a bundle map of the other bundle structure π_* on TE, which it could map to π over τ_M. For although $\pi \circ D = \tau_M \circ \pi_*$, in general D will not be linear on the fibres of π_*. If D *is* linear on these fibres, then the horizontal distribution $\mathscr{H}$ is called a *connexion* on π, and D is called the *connexion map*. If $f: M \to E$ is a cross-section of π, and $X \in T_x$, the covariant derivative $\nabla_X f \in E_x$ with respect to this connexion is defined by

$$\nabla_X f = D(f_* X). \tag{2.3}$$

If $X: M \to TM$ is a vector field on M, then $\nabla_X f$ is also a cross-section of π, and (2.3) becomes

$$\nabla_X f = D \circ f_* \circ X. \tag{2.4}$$

We now tie this up with the concept of a connexion as used in elementary tensor calculus. This is done through the notion of parallel displacement. If γ is a curve in M through x and y, and if $v \in E_x$, then there is a unique lift Γ of γ through v whose tangent is everywhere horizontal. It is called the horizontal lift of γ through v, and if w is the point of Γ above y, we say that w is obtained from v by parallel displacement along γ. The linearity of D on the fibres of π_* ensures that parallel displacement maps E_x linearly onto E_y. By identifying the parallel displacement of tensors in tensor calculus with the parallel displacement of the fibres in a tensor bundle in this sense, we see that an affine connexion on M in the sense of tensor calculus provides a bundle connexion on each tensor bundle $T^r_s(M)$ in the above sense. We can also put (2.3) in a more familiar form. Let γ be a curve in M with $\gamma'(0) = X$, and let $v(t) \in E_x$ be obtained from $f \circ \gamma(t)$ by parallel displacement back along γ. Then

$$\nabla_X f = [\mathrm{d}v/\mathrm{d}t]_{t=0}. \tag{2.5}$$

This agrees with the definition used in tensor calculus if E is a tensor bundle over M.

The final construction that we shall need is that of induced connexions. Let $\pi: E \to M$ and $\rho: F \to N$ be vector bundles of the same dimension, and let $\alpha: F \to E$ be a bundle map which maps fibres isomorphically. If $\mathscr{H}$ is a connexion on π, then $\alpha_*^{-1}\mathscr{H}$ is a connexion on ρ, said to be induced from $\mathscr{H}$ by α. It is the unique connexion on ρ such that α_* maps the horizontal subspaces of TF linearly onto horizontal subspaces of TE. Note that α_* always preserves vertical subspaces. The corresponding connexion maps D_E and D_F are related by

$$\alpha \circ D_F = D_E \circ \alpha_*. \tag{2.6}$$

In particular, we can apply this to an induced bundle $\beta^{\#}\pi$. Let Y be a vector field on N, h be a cross-section of $\beta^{\#}\pi$ and $\bar{h}$ be defined by (2.1). We shall write $\nabla_Y \bar{h}$ for the map $N \to E$ representing $\nabla_Y h$ under the correspondence (2.1). Then from (2.1), (2.4) and (2.6) we see that

$$\nabla_Y \bar{h} = D_E \circ \bar{h}_* \circ Y, \tag{2.7}$$

which involves only the connexion map on π.

We can now apply the above to the case which will be of most importance in our theory of moments. Let M be a pseudo-Riemannian manifold, and take π to be the tensor bundle over M of type (r, s). Considering $\tau: TM \to M$ simply as a mapping of manifolds, we form the induced bundle $\tau^{\#}\pi$ over TM. On this we put the connexion induced by $\pi^{\#}\tau$ from the Riemannian connexion on M. This gives us enough structure to enable us to define a covariant derivative of a tensor field Φ on TM over τ. It is given by (2.7), which defines $\nabla_P \Phi$ if $P: TM \to TTM$ is a vector field on TM.

We next evaluate $\nabla_P \Phi$ in terms of the ordinary covariant derivative on M. Let Γ be an orbit (integral curve) of P, i.e. a curve in TM such that $\Gamma' = P \circ \Gamma$, and put $\gamma = \tau \circ \Gamma$. Then γ is a curve in M, and Γ is a lift of γ to TM. Hence there is a vector field X on M such that $\Gamma = X \circ \gamma$, X being uniquely defined only along γ. This implies $\Gamma' = X_* \circ \gamma'$, and so on using (2.7) and (2.4), we get

$$(\nabla_P \Phi) \circ \Gamma = \nabla_{\gamma'}(\Phi \circ X). \tag{2.8}$$

As $\Phi \circ X$ is a vector field on M, the final covariant derivative in (2.8) is the ordinary one on M. As there exists an orbit of P through every point of TM, this thus completely determines $\nabla_P \Phi$ in the required form.

Since the construction of TTM is rather complicated, we now introduce an alternative notation which only involves the manifold TM. Let D be the connexion map of the Riemannian connexion on τ. We saw above that if $Q \in T_X(TM)$, then the map $Q \mapsto (\tau_* Q, DQ)$ is an isomorphism of $T_X(TM)$ onto $T_x(M) \oplus T_x(M)$, where $x = \tau(X)$. We use this to write $\nabla_P \Phi(X)$ as $\nabla_{(A,B)}\Phi(X)$,

where $$A := \tau_* P_X \quad \text{and} \quad B := DP_X.$$

Note that it takes the triple (A, B, X) of tangent vectors to M at x to specify a value of P completely. This is why we write the field point X explicitly in $\nabla_{(A,B)}\Phi(X)$. However, we can denote the field $\nabla_P \Phi$ unambiguously by $\nabla_{(A,B)}\Phi$ if we let A and B become vector fields on TM over τ. We need to put

$$A = \tau_* \circ P, \quad B = D \circ P, \tag{2.9}$$

so that $A_X = \tau_* P_X$, $B_X = DP_X$ as before. We see that $\tau \circ A = \tau \circ B = \tau$, so that A, B are indeed vector fields on TM over τ in the sense defined above.

Let us now find local coordinate expressions for these results. If $\{z^\lambda\}$ is a local coordinate system on M, the corresponding coordinates in TM of a point $X \in TM$ are the coordinates z^λ of the base point $z = \tau(X)$ together with the components X^λ of X with respect to $\{z^\lambda\}$. The map $\Phi: TM \to E$ will thus be a function of the variables (z^λ, X^λ), and since $\pi \circ \Phi = \tau$, its value at (z^λ, X^λ) will be a tensor of type (r, s) on M at z. For the moment we shall suppress the tensor indices on Φ. The fields A and B of (2.9) will be similar functions, their values being vectors at z. Since $\nabla_{(A,B)}\Phi$ is linear in A and B, we can define two covariant derivative operators $\nabla_{\lambda *}$ and $\nabla_{*\lambda}$ by putting

$$\nabla_{(A,B)}\Phi = A^\lambda \nabla_{\lambda *}\Phi + B^\lambda \nabla_{*\lambda}\Phi. \tag{2.10}$$

Then $\nabla_{\lambda *}\Phi$ and $\nabla_{*\lambda}\Phi$ will be tensor fields on TM over τ of type $(r, s+1)$, so that both operators raise the covariant degree of the field by one, as does the usual covariant derivative ∇_λ.

In the notation of (2.8), let $(z^\lambda(t), X^\lambda(t))$ be the coordinates of $\Gamma(t) \in TM$. Then the coordinates of $\gamma(t) \in M$ are $z^\lambda(t)$. But from (2.4) and (2.9) we have

$$A \circ \Gamma = \tau_* \circ \Gamma' = \gamma', \quad B \circ \Gamma = D \circ X_* \circ \gamma' = \nabla_{\gamma'} X. \tag{2.11}$$

On expressing these in component form, we thus get

$$A^\lambda = \frac{\mathrm{d}z^\lambda}{\mathrm{d}t}, \quad B^\lambda = \frac{\mathrm{d}X^\lambda}{\mathrm{d}t} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}z^\mu}{\mathrm{d}t}X^\nu. \tag{2.12}$$

To write out (2.8) in component form, let us take $r = s = 1$, so that Φ has index form $\Phi^\lambda{}_{.\mu}$. Then (2.8) gives

$$\nabla_{(A,B)}\Phi^\lambda{}_{.\mu} = \mathrm{d}\Phi^\lambda{}_{.\mu}/\mathrm{d}t + \Gamma^\lambda_{\kappa\nu}A^\kappa\Phi^\nu{}_{.\lambda} - \Gamma^\nu_{\kappa\mu}A^\kappa\Phi^\lambda{}_{.\nu}. \tag{2.13}$$

But on using (2.12) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^\lambda{}_{.\mu} = A^\kappa\left\{\frac{\partial}{\partial z^\kappa}\Phi^\lambda{}_{.\mu} - \Gamma^\rho_{\kappa\nu}X^\nu\frac{\partial}{\partial X^\rho}\Phi^\lambda{}_{.\mu}\right\} + B^\kappa\frac{\partial}{\partial X^\kappa}\Phi^\lambda{}_{.\mu}. \tag{2.14}$$

On putting this into (2.13) and comparing the result with (2.10), we then find that

$$\nabla_{\kappa *}\Phi^\lambda{}_{.\mu} = \frac{\partial}{\partial z^\kappa}\Phi^\lambda{}_{.\mu} - \Gamma^\rho_{\kappa\nu}X^\nu\frac{\partial}{\partial X^\rho}\Phi^\lambda{}_{.\mu} + \Gamma^\lambda_{\kappa\nu}\Phi^\nu{}_{.\mu} - \Gamma^\nu_{\kappa\mu}\Phi^\lambda{}_{.\nu} \tag{2.15}$$

and

$$\nabla_{*\kappa}\Phi^\lambda{}_{.\mu} = \frac{\partial}{\partial X^\kappa}\Phi^\lambda{}_{.\mu}. \tag{2.16}$$

We see that $\nabla_{\kappa *}\Phi^\lambda{}_{.\mu}$ differs from the ordinary form of a covariant derivative only by the presence of the extra X-derivative term. Clearly this will also be so for arbitrary r and s. As a simple but important illustration of (2.15), we note that

$$\nabla_{\kappa *}X^\lambda = 0. \tag{2.17}$$

Our next development involves integrating over the tangent spaces $T_z(M)$. Let η be the volume element on $T_z(M)$. In integration theory, this is an m-form on the $2m$-dimensional manifold TM, but one can equally well consider it to be the usual coordinate expression

$$\mathrm{D}X := \sqrt{|g(z)|}\,\mathrm{d}X^1 \ldots \mathrm{d}X^m. \tag{2.18}$$

We shall write η if using other invariant notation, and $\mathrm{D}X$ if using coordinate expressions. Then if Φ has compact support on TM, we may define a tensor field $\phi : M \to E$ by

$$\phi(z) := \int_{T_z(M)} \Phi\eta. \tag{2.19}$$

To state our main result concerning such integrals, we first define the horizontal lift A^h of a vector field A on M. This is the unique horizontal vector field on TM satisfying $\tau_* \circ A^h = A \circ \tau$. In the representation (2.9) of vector fields on TM, A^h corresponds to $(A \circ \tau, 0)$. We now prove that

$$\nabla_A\phi(z) = \int_{T_z} \nabla_{A^h}\Phi\eta, \quad \text{i.e.} \quad \nabla_\kappa\phi(z) = \int_{T_z} \nabla_{\kappa *}\Phi\,\mathrm{D}X, \tag{2.20}$$

the second form being the corresponding local coordinate expression.

Let γ be an orbit of A, and let $\beta_t: T_{\gamma(0)}(M) \to T_{\gamma(t)}(M)$ be the isomorphism induced by parallel displacement of the fibres of τ along γ. For each $X \in T_{\gamma(0)}(M)$, let βX be the curve in TM defined by $(\beta X)_t = \beta_t X$. This will be an orbit of A^h, and hence will satisfy

$$(\beta X)' = A^h \circ (\beta X). \tag{2.21}$$

Now from (2.19) we have

$$\phi \circ \gamma(t) = \int_{T_{\gamma(0)}} \Phi \circ \beta_t \eta, \tag{2.22}$$

since β_t is an isometry, and hence on differentiating with respect to t using (2.21), we get

$$\phi_* \circ \gamma'(t) = \int_{T_{\gamma(0)}} \Phi_* \circ A^h \circ \beta_t \eta = \int_{T_{\gamma(t)}} \Phi_* \circ A^h \eta. \tag{2.23}$$

But $\gamma' = A \circ \gamma$. Hence, since γ is an arbitrary orbit of A, (2.23) implies

$$\phi_* \circ A(z) = \int_{T_z} \Phi_* \circ A^h \eta \tag{2.24}$$

for all z. Equation (2.20) then follows, as required, on applying D_E to both sides and using (2.7) and (2.4).

Still considering Φ to be of compact support, we define the Fourier transform $\tilde{\Phi}$ of Φ by taking the ordinary Fourier transform separately on each tangent space. Thus if $k \in T_z(M)$, we define

$$\tilde{\Phi}(k) := \int_{T_z} \Phi(X) \exp(\mathrm{i}k.X)\, \mathrm{D}X, \tag{2.25}$$

where $k.X := g_{\kappa\lambda}(z)\, k^\kappa X^\lambda$. We shall also write $\tilde{\Phi}$ as $F\Phi$. Clearly, $\tilde{\Phi}$ is also a tensor field on TM over τ of the same type as Φ. A slight modification of the above proof shows that

$$\nabla_{A^h} F\Phi = F\nabla_{A^h}\Phi, \quad \text{i.e.} \quad \nabla_{\kappa *} F\Phi = F\nabla_{\kappa *}\Phi. \tag{2.26}$$

We also have, from (2.16) and (2.25), that

$$F\nabla_{*\lambda}\Phi = -\mathrm{i}k_\lambda F\Phi, \quad F(X_\lambda \Phi) = -\mathrm{i}\nabla_{*\lambda} F\Phi \tag{2.27}$$

as for ordinary Fourier transforms.

3. The correspondence between bitensors and tensor fields over τ

In the preceding section we developed some properties of tensors on TM over τ. We shall now relate these to a more familiar type of induced tensor field. Let p_1 and p_2 be the natural projections of the product manifold $M \times M$ onto its first and second elements respectively. Then a tensor field on $M \times M$ over p_1 (resp. p_2) is just a field of two-point tensors, or bitensors, on M with scalar character at its second (resp. first) argument point. By using the exponential map $\mathrm{Exp}: TM \to M$ of the Riemannian connexion on M, we shall develop one-to-one correspondences between tensor fields over τ and those over p_1 and p_2.

We may consider M as a submanifold of TM by identifying each point $z \in M$ with the zero vector of $T_z(M)$. It is then well known that there exists a neighbourhood N of M in TM such that the map $\delta: N \to M \times M$ given by

$$\delta(X) = (\tau(X), \mathrm{Exp}\, X) \tag{3.1}$$

is a diffeomorphism of N onto its image. A proof may be found in Kobayashi & Nomizu (1963, ch. III, §8) and other standard texts. As $p_1 \circ \delta = \tau$, if $\hat{\Phi}$ is a tensor field on $M \times M$ over p_1 then

$$\Phi := \hat{\Phi} \circ \delta \tag{3.2}$$

is a tensor field on N over τ. Since δ is a diffeomorphism, if we restrict $\hat{\Phi}$ to $\delta(N) \subset M \times M$ then this correspondence between Φ and $\hat{\Phi}$ is one-to-one.

The covariant derivative of $\hat{\Phi}$ is, by (2.7), defined with respect to a vector field on $M \times M$. But if $(z, x) \in M \times M$, there is a natural isomorphism $T_{(z,x)}(M \times M) \cong T_z(M) \oplus T_x(M)$. If, under this, $Q \mapsto (A, B)$, we shall write $\nabla_Q \hat{\Phi}$ as $\nabla_{(A,B)} \hat{\Phi}$. Note that A and B will, in general, be vectors at different points of M, in contrast to the case for $\nabla_{(A,B)} \Phi(X)$ defined in §2, and that there is no need to explicitly state the field point (z, x) as it is determined by A and B. If A and B are both vector fields on M, then $\nabla_{(A,B)} \hat{\Phi}$ is a tensor field on $M \times M$ over p_1. In local coordinates, we shall write

$$\nabla_{(A,B)} \hat{\Phi}(z, x) = A^\kappa(z) \nabla_\kappa \hat{\Phi} + B^\alpha(x) \nabla_\alpha \hat{\Phi}, \tag{3.3}$$

indices on $\hat{\Phi}$ being suppressed. Here, and throughout, we are using the convention that $\alpha, \beta, \ldots$ will denote tensor indices at x and $\kappa, \lambda, \ldots$ similarly at z. This provides a sufficient distinction between the derivatives of $\hat{\Phi}$ at the two points that no further distinction, such as was needed in (2.10), is necessary.

We now wish to relate the covariant derivative of $\hat{\Phi}$ to that of Φ. From (3.2) we have

$$\Phi_* = \hat{\Phi}_* \circ \delta_*.$$

If we use this in (2.7) and let $P \in T_X(TM)$, we immediately see that

$$\nabla_P \Phi = \nabla_{\delta_* P} \hat{\Phi}. \tag{3.4}$$

Now let

$$A = \tau_* P, \quad B = DP, \quad C = \mathrm{Exp}_* P, \quad (z, x) = \delta(X). \tag{3.5}$$

Then on using (3.1), we can express (3.4) as

$$\nabla_{(A,B)} \Phi = \nabla_{(A,C)} \hat{\Phi}. \tag{3.6}$$

It remains only to express C in terms of A and B. For this we use local coordinate expressions, with the conventions for bitensor notation given in appendix 1. The exponential map is related to the world function biscalar σ by

$$\mathrm{Exp}_z X = x \quad \text{if} \quad X^\kappa = -\sigma^\kappa(z, x). \tag{3.7}$$

We apply (3.6) with $E = TM$ and Φ being the identity map on TM. From (3.1), (3.2) and (3.7) we thus see that the corresponding $\hat{\Phi}$ is

$$\hat{\Phi}^\kappa(z, x) = -\sigma^\kappa(z, x). \tag{3.8}$$

But $\nabla_{(A,B)} \Phi = D\Phi_* P = B$. On putting this into (3.6) and using (3.3) and (3.8), we can solve for C^α to give

$$C^\alpha = K^\alpha_{.\kappa} A^\kappa + H^\alpha_{.\kappa} B^\kappa, \tag{3.9}$$

where

$$K^\alpha_{.\kappa} := -\overset{-1}{\sigma}{}^\alpha_{.\lambda} \sigma^\lambda_{.\kappa}, \quad H^\alpha_{.\kappa} := -\overset{-1}{\sigma}{}^\alpha_{.\kappa}. \tag{3.10}$$

These definitions agree with those given in I by (I, 3.9). On substituting (3.9) back into (3.6) and using (3.3) and (2.10), we obtain the desired relations

$$\nabla_{\lambda *} \Phi = \nabla_\lambda \hat{\Phi} + K^\alpha_{.\lambda} \nabla_\alpha \hat{\Phi}, \quad \nabla_{*\lambda} \Phi = H^\alpha_{.\lambda} \nabla_\alpha \hat{\Phi}, \tag{3.11}$$

the indices on Φ and $\hat{\Phi}$ being suppressed.

The correspondence between Φ and $\hat{\Phi}$ is fundamental to the development of the theory of moments. It is frequently convenient not to distinguish between them notationally, as the appropriate form is almost always clear from the context. We shall then write (3.11) with the circumflexes omitted, as the types of derivative symbol used provide sufficient distinction between the two forms. We shall, however, preserve the distinction for the remainder of the present section.

The correspondence $\Phi \mapsto \hat{\Phi}$ associates with Φ a tensor field $\hat{\Phi}$ on $\delta(N) \subset M \times M$ over p_1. There is a second correspondence that we shall need, also generated by the exponential map, which associates with Φ a tensor field ϕ on $\delta(N)$ over p_2. Let $z \in M$. Then since Exp_z maps

$$N_z := T_z(M) \cap N$$

diffeomorphically onto its image in M, the results of § 2 show that its derivative can be extended to give a bundle map $(\mathrm{Exp}_z)_A$ over Exp_z of the corresponding tensor bundles of type (r, s). The second of these bundles is just the portion of $E = T^r_s(M)$ over $\mathrm{Exp}\, N_z$, while the first is simply represented using the canonical isomorphism

$$T^r_s(T_z(M)) \cong T_z(M) \oplus (T^r_s)_z(M). \tag{3.12}$$

We then define ϕ at $(z, x) := \delta(X)$ by

$$\phi(z, x) := (\mathrm{Exp}_z)_A (X, \Phi(X)) \subset E. \tag{3.13}$$

This gives $\pi \circ \phi(z, x) = \mathrm{Exp}\, X = p_2(z, x)$, so that ϕ is indeed a tensor field on $M \times M$ over p_2, as required.

Since $(\mathrm{Exp}_z)_A$ is a diffeomorphism, the corespondence between the fields ϕ and Φ is one-to-one. With an abuse of notation, we shall write

$$\phi = \mathrm{Exp}_A \Phi \quad \text{on} \quad \delta(N), \quad \Phi = \mathrm{Exp}^A \phi \quad \text{on} \quad N. \tag{3.14}$$

This cannot cause confusion as Exp_A and Exp^A are not meaningful in the notation of § 2, since $\mathrm{Exp}: TM \to M$ is not a diffeomorphism. We shall make much use of this correspondence, and the convenience of a simple notation for it, reminiscent of the true definition (3.13), outweighs the disadvantage of a slight notational inconsistency. The coordinate form of (3.13) is easily seen, from (2.2), (3.7) and (3.10), to be

$$\phi^{\alpha\ldots}{}_{\beta\ldots}(z, x) = H^{\alpha}_{\cdot\kappa} \ldots (-\sigma_{;\beta}^{;\lambda}) \ldots \Phi^{\kappa\ldots}{}_{\lambda\ldots}(X). \tag{3.15}$$

With the use of (3.2), this can also be expressed as a relation between the two bitensor fields ϕ and $\hat{\Phi}$,

$$\phi^{\alpha\ldots}{}_{\beta\ldots}(z, x) = H^{\alpha}_{\cdot\kappa} \ldots (-\sigma_{;\beta}^{;\lambda}) \ldots \hat{\Phi}^{\kappa\ldots}{}_{\lambda\ldots}(z, x), \tag{3.16}$$

showing that one can be obtained from the other by using suitable propagators to transfer the tensor character from z to x or vice versa.

A simple interpretation can be given of the relationship between ϕ and Φ in terms of normal coordinates. Suppose we start with an arbitrary coordinate system $\{x^\alpha\}$, choose a fixed point z, and then set up around z the (unique) normal coordinate system $\{x^{\alpha'}\}$ such that $\partial x^{\alpha'}/\partial x^\alpha = \delta^{\alpha'}_\alpha$ at z. Then if $X \in T_z(M)$, the components of $\Phi(X)$ in the original coordinate system are precisely equal to the components of $\phi(z, x)$ in this normal coordinate system if x is the point whose normal coordinates are the components of X. In the special case where ϕ is independent of z, Φ gives a way of simultaneously describing ϕ in normal coordinate systems about every point of the manifold.

There is an important relation between the (ϕ, Φ) correspondence and Lie differentiation. If X is a vector field on a manifold M_1, we denote Lie differentiation with respect to X by L_X. Then if $\theta: M_1 \to M_2$ is a diffeomorphism, we have

$$\theta_A \circ L_X = L_{\theta_* X} \circ \theta_A. \tag{3.17}$$

Now suppose that Λ and Φ are respectively vector and tensor fields on TM over τ. By restricting Λ and Φ to $T_z(M)$, in view of (3.12) they become ordinary vector and tensor fields on this vector space, so that $L_\Lambda \Phi$ is similarly well defined on $T_z(M)$. By letting z vary over M, we can thus consider $L_\Lambda \Phi$ as a tensor field on TM over τ. Now let $\phi = \mathrm{Exp}_A \Phi$ and $\lambda = \mathrm{Exp}_A \Lambda$. Define $L_\lambda \phi$, similarly to the above, as a tensor field on $M \times M$ over p_2 by considering separately each fixed value of z. Then on applying (3.17) with $\theta = \mathrm{Exp}_z$, we find that

$$L_\lambda \phi = \mathrm{Exp}_A L_\Lambda \Phi \quad \text{on} \quad \delta(N). \tag{3.18}$$

While this is a perfectly acceptable definition of $L_\lambda \phi$ when the bitensors λ and ϕ are taken as having scalar character at the same point z, it is not the only possible interpretation, and it is perhaps not the most natural one. If v is a vector field on M and ϕ is any bitensor on M, possibly with tensorial character at both argument points, then $L_v \phi$ is most naturally defined by letting L_v act separately at each point. For example,

$$L_v \phi^{\alpha}_{\cdot\kappa} = v^\beta \partial_\beta \phi^{\alpha}_{\cdot\kappa} - \phi^{\beta}_{\cdot\kappa} \partial_\beta v^\alpha + v^\lambda \partial_\lambda \phi^{\alpha}_{\cdot\kappa} + \phi^{\alpha}_{\cdot\lambda} \partial_\kappa v^\lambda, \tag{3.19}$$

where the first two terms on the right hand side come from considering z as fixed and differentiating at x, while the last two terms fix x and differentiate at z. If now v is replaced by a bitensor field $v^\alpha(y, x)$, with scalar character at y, then $L_v \phi$ is most naturally considered as a *three*-point tensor, with scalar character at y and defined for each fixed y as a bitensor as in (3.19). The case when y and z coincide is then treated as a coincidence limit of this three-point tensor as $y \to z$. If ϕ has scalar character at z, we see that this agrees with the previous interpretation of $L_\lambda \phi$ used in (3.17) provided

$$\lim_{x \to z} \lambda^\alpha(z, x) = 0 \tag{3.20}$$

identically. We shall adopt this latter interpretation for Lie derivatives involving bitensors, and hence (3.20) must be added as an additional condition for the validity of (3.18).

In the notation of (3.18), a third meaningful Lie derivative is $L_\lambda \hat{\Phi}$, where

$$\lambda^\alpha = \lambda^\alpha(y, x), \quad \hat{\Phi} = \hat{\Phi}(z, x).$$

Let us again take the limit $y \to z$. Since the result is a tensor field on $\delta(N)$ over p_1, we can associate with it a tensor field on N over τ by (3.2). We shall denote this by $L_\lambda \Phi$, so that

$$L_\lambda \Phi := (L_\lambda \hat{\Phi}) \circ \delta, \tag{3.21}$$

thus giving meaning to L_λ, as well as L_Λ, acting on Φ. As a companion result to (3.18), we now investigate under what conditions

$$L_\lambda \mathrm{Exp}_A \Phi = \mathrm{Exp}_A L_\lambda \Phi \quad \text{on} \quad \delta(N). \tag{3.22}$$

From (3.16), we see that (3.22) will hold for all Φ if

$$L_\lambda H^{\alpha}_{\cdot\kappa} = 0 \quad \text{and} \quad L_\lambda \sigma^{\kappa}_{\cdot\alpha} = 0. \tag{3.23}$$

The first of these follows from the second by (3.10). So since we also have

$$L_\lambda \sigma^\kappa_{\cdot\alpha} = \partial_\alpha L_\lambda \sigma^\kappa, \tag{3.24}$$

it follows that (3.22) will hold provided

$$L_\lambda \sigma^\kappa \equiv \lambda^\alpha \sigma^\kappa_{\cdot\alpha} + \lambda^\mu \sigma^\kappa_{\cdot\mu} - \sigma^\mu \nabla_\mu \lambda^\kappa = 0. \tag{3.25}$$

We can choose λ^μ and $\nabla_\mu \lambda^\kappa$ arbitrarily as they are independent of x, and then solve (3.25) for $\lambda^\alpha(z, x)$ for all $x \neq z$. On using (3.10), we obtain $\lambda^\alpha = \xi^\alpha$, where

$$\xi^\alpha(z, x) := K^\alpha_{\cdot\kappa} A^\kappa + H^\alpha_{\cdot\kappa} \sigma_\lambda B^{\kappa\lambda} \tag{3.26}$$

and $A^\kappa(z)$, $B^{\kappa\lambda}(z)$ are arbitrary tensor fields. For each fixed choice of z, this gives a 20-parameter family of vector fields which was seen, in § 3 of I, to include all the Killing vectors of the manifold, as is to be expected from the geometrical interpretation of (3.25). Note that unless $A^\kappa = 0$, ξ^α does not satisfy (3.20), so that neither one of (3.18) and (3.22) necessarily implies the other.

From (3.11), (3.21) and (3.26) we see that the coordinate expression for $L_\xi \Phi$ is given by

$$L_\xi \Phi^{\kappa\cdots}{}_{\lambda\ldots} = A^\mu \nabla_{\mu *} \Phi^{\kappa\cdots}{}_{\lambda\ldots} + B^{\mu\nu} X_\mu \nabla_{*\nu} \Phi^{\kappa\cdots}{}_{\lambda\ldots} + B^\kappa_{\cdot\mu} \Phi^{\mu\cdots}{}_{\lambda\ldots} + \ldots - B^\mu_{\cdot\lambda} \Phi^{\kappa\cdots}{}_{\mu\ldots} - \ldots. \tag{3.27}$$

Taken together with (2.26) and (2.27), this shows that L_ξ commutes with Fourier transformation provided

$$B_{(\kappa\lambda)} = 0. \tag{3.28}$$

4. Generating functions for the reduced moments

Having considered the mathematical techniques that we shall need to handle tensor fields on TM over τ, we now introduce the important fields of this type that play a central role in the theory. These are the generating functions for the moments of $T^{\alpha\beta}$ and J^α. We first give the symmetry and orthogonality properties that we shall require of the moments, without inquiring too closely how they are to be related to the corresponding fields. From the moments we construct the generating functions, and then we investigate the properties possessed by these functions in virtue of the imposed conditions. Only after this shall we consider how to relate the moments to the fields. For the initial stages our guideline will be the results obtained in II for the simpler case of J^α.

We first choose a base line l. This is an arbitrary C^∞ timelike world line which may be considered as representing an observer, and it provides the (moving) origin to which the moments are referred. Additionally, we choose an arbitrary C^∞ field n^λ of timelike unit vectors along l, which is to be the instantaneous four-velocity of the reference frame used in constructing the moments. It may seem most natural to choose n^λ tangent to l, but for reasons discussed in I there are considerable advantages to be gained by having n^λ arbitrary. We parametrize l as $z^\lambda(s)$, so that indices $\kappa, \lambda, \ldots$ are used for tensors along l, and x^α is used as an arbitrary field point. It is also convenient not to insist that s be the proper time along l, so that the velocity vector $v^\lambda := \mathrm{d}z^\lambda/\mathrm{d}s$ is not necessarily a unit vector. Later, we shall see that the most natural condition on s is to take $n_\lambda v^\lambda = 1$, but for the present we also leave s arbitrary.

First consider J^α. In any set of moments of J^α in the sense defined in II, the 2^n-pole moment tensor $m^{\lambda_1\cdots\lambda_n\mu}(s)$ satisfies

$$m^{\lambda_1\cdots\lambda_n\mu} = m^{(\lambda_1\cdots\lambda_n)\mu} \quad \text{for} \quad n \geqslant 2. \tag{4.1}$$

To discuss general properties of tensor symmetries, we shall use the representation theory of the symmetric group as expressed, for example, in Weyl (1946). We write $[n_1, n_2, \ldots, n_r]$, where $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$, to denote the irreducible symmetry described by the Young diagram of the partition $(n_1, n_2, \ldots, n_r)$. Then if $n \geqslant 1$, (4.1) corresponds to the reducible symmetry

$$[n] \otimes [1] = [n, 1] \oplus [n+1], \tag{4.2}$$

where the parts with symmetries $[n, 1]$ and $[n+1]$ may be taken as

$$m^{\lambda_1 \ldots \lambda_{n-1}[\lambda_n \mu]} \quad \text{and} \quad m^{(\lambda_1 \ldots \lambda_n \mu)} \tag{4.3}$$

respectively. It was shown in II that the appropriate additional conditions to impose to obtain the reduced moments of J^α when (1.31) is satisfied are

$$m^{(\lambda_1 \ldots \lambda_n \mu)} = 0 \quad \text{for} \quad n \geqslant 1 \tag{4.4}$$

and

$$n_{\lambda_1} m^{\lambda_1 \ldots \lambda_{n-1}[\lambda_n \mu]} = 0 \quad \text{for} \quad n \geqslant 2. \tag{4.5}$$

Condition (4.4) gives $m^{\cdots}$ the irreducible symmetry $[n, 1]$, while (4.5) gives

$$n_{\lambda_1} m^{\lambda_1 \ldots \lambda_n \mu} \tag{4.6}$$

the irreducible symmetry $[n]$, i.e. makes it totally symmetric.

We now use this as a guide to suggest appropriate conditions for the moments of $T^{\alpha\beta}$. In any set of moments of $T^{\alpha\beta}$, the 2^n-pole moment tensor should satisfy

$$I^{\lambda_1 \ldots \lambda_n \mu\nu} = I^{(\lambda_1 \ldots \lambda_n)(\mu\nu)} \quad \text{for} \quad n \geqslant 0. \tag{4.7}$$

If $n = 0$, this is already irreducible, while if $n = 1$ it has the reducible symmetry

$$[1] \otimes [2] = [2, 1] \oplus [3]. \tag{4.8}$$

But for the general case $n \geqslant 2$, it has the symmetry

$$[n] \otimes [2] = [n, 2] \oplus [n+1, 1] \oplus [n+2]. \tag{4.9}$$

Analogy with J^α suggests that we require the $[n+2]$ part to vanish for $n \geqslant 1$, and the $[n+1, 1]$ part also to vanish when $n \geqslant 2$. This leaves the moments of all orders with irreducible symmetries, namely $[n, 2]$ for $n \geqslant 2$ and $[2, 1]$ for $n = 1$, and is achieved by requiring

$$I^{(\lambda\mu\nu)} = 0, \tag{4.10}$$

and

$$I^{(\lambda_1 \ldots \lambda_n \mu)\nu} = 0 \quad \text{for} \quad n \geqslant 2. \tag{4.11}$$

Now it follows from (4.7) and (4.11) that

$$I^{\lambda_1(\lambda_2 \ldots \lambda_n \mu\nu)} = 0 \quad \text{if} \quad n \geqslant 2. \tag{4.12}$$

Hence

$$n_{\lambda_1} I^{\lambda_1 \ldots \lambda_n \mu\nu}, \tag{4.13}$$

for $n \geqslant 2$, has the reducible symmetry $[n-1, 2] \oplus [n, 1]$, as (4.12) gives the vanishing of the $[n+1]$ part which it would also have had if only (4.7) were satisfied. Again by analogy with the case for J^α, we impose the vanishing of the $[n-1, 2]$ part of (4.13) by requiring

$$n_{\lambda_1} I^{\lambda_1 \ldots \lambda_{n-2}[\lambda_{n-1}[\lambda_n \mu]\nu]} = 0 \quad \text{for} \quad n \geqslant 3, \tag{4.14}$$

leaving (4.13) with the irreducible symmetry $[n, 1]$. No orthogonality condition is imposed on $I^{\lambda\mu\nu}$, as $n_\lambda I^{\lambda\mu\nu}$ is already irreducible.

The plausibility of these conditions is increased by the fact that for a flat spacetime in which we choose $n^\lambda = v^\lambda$, they are satisfied by the reduced moments obtained by Dixon (1967) by performing the reduction process explicitly. Their real justification, however, is simply that we shall see that they work. For the moment they are just a hypothesis that we shall investigate.

The corresponding moment generating functions are tensor fields on TM over τ, although they are defined only on the portion of TM over the base line l. They are defined by

$$\tilde{m}^\lambda(s,k) := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\kappa_1} \dots k_{\kappa_n} m^{\kappa_1 \dots \kappa_n \lambda}(s) \tag{4.15}$$

and

$$\tilde{I}^{\lambda\mu}(s,k) := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\kappa_1} \dots k_{\kappa_n} I^{\kappa_1 \dots \kappa_n \lambda\mu}(s) \tag{4.16}$$

so that $\tilde{I}^{\lambda\mu}$ is symmetric. The conditions (4.1) and (4.7) are then precisely those necessary for the generating functions to uniquely determine the moments. The additional condition (4.4) is equivalent to

$$k_\lambda \tilde{m}^\lambda = k_\lambda m^\lambda, \tag{4.17}$$

while (4.11) is similarly equivalent to

$$k_\lambda \tilde{I}^{\lambda\mu} = k_\lambda I^{\lambda\mu} - \mathrm{i} k_\kappa k_\lambda I^{\kappa\lambda\mu}, \tag{4.18}$$

leaving (4.10) as the only remaining symmetry condition needing separate consideration.

To express the orthogonality conditions (4.5) and (4.14) in terms of the generating functions, it is convenient to refer the tensors to an orthonormal tetrad system $\underset{\kappa}{e}^\lambda(s)$ defined along l in which $\underset{0}{e}^\lambda(s) = n^\lambda(s)$. Equalities holding only with this special choice of basis will be denoted by $\overset{*}{=}$. It is then easily seen that (4.5) and (4.14) are respectively equivalent to $\nabla_{*[\lambda}\tilde{m}_{\mu]}$ and $\nabla_{*[\kappa[\lambda}\tilde{I}_{\mu]\nu]}$ being independent of k^0. We are here using the notation (2.16) and writing $\nabla_{*\kappa\lambda} := \nabla_{*\kappa}\nabla_{*\lambda}$ for repeated derivatives.

We now use this to investigate the k^0-dependence of $\tilde{m}^\lambda$ and $\tilde{I}^{\lambda\mu}$ themselves, starting first with $\tilde{m}^\lambda$. On multiplying (4.4) by $n_{\lambda_1} n_{\lambda_2}$ and using (4.1) and (4.5), we obtain

$$n_{\lambda_1} n_{\lambda_2} m^{\lambda_1 \dots \lambda_n \mu} = 0 \quad \text{for} \quad n \geqslant 2, \tag{4.19}$$

and hence from (4.15) that

$$\tilde{m}^\lambda(s,k) \overset{*}{=} A^\lambda(s,\boldsymbol{k}) + k^0 B^\lambda(s,\boldsymbol{k}), \tag{4.20}$$

where $\boldsymbol{k} := (k^1, k^2, k^3)$, for suitable functions A^λ and B^λ independent of k^0. The requirement that $\nabla_{*[\lambda}\tilde{m}_{\mu]}$ be independent of k^0 is then equivalent to

$$\nabla_{*[\lambda} B_{\mu]} = 0, \tag{4.21}$$

and thus to the existence of a scalar function $B(s,k)$ such that

$$B_\mu = \nabla_{*\mu} B. \tag{4.22}$$

But from (4.17) and (4.20) we see that $B_0 \overset{*}{=} 0$. Hence by (4.22), B is independent of k^0, so that

$$\tilde{m}_\lambda(s,k) \overset{*}{=} A_\lambda(s,\boldsymbol{k}) + k^0 \nabla_{*\lambda} B(s,\boldsymbol{k}), \tag{4.23}$$

a result derived in II by a different method.

To obtain the analogous result for $\tilde{I}_{\lambda\mu}$, we first deduce from (4.14) and the symmetry conditions that

$$n_{\lambda_1} n_{\lambda_2} n_{\lambda_3} I^{\lambda_1 \dots \lambda_n \mu\nu} = 0 \quad \text{if} \quad n \geqslant 3, \tag{4.24}$$

and hence from (4.16) that $\tilde{I}^{\mu\nu}$ can be expressed in the form

$$\tilde{I}^{\mu\nu}(s,k) \stackrel{*}{=} A^{\mu\nu}(s,\boldsymbol{k}) + k^0 B^{\mu\nu}(s,\boldsymbol{k}) + (k^0)^2 C^{\mu\nu}(s,\boldsymbol{k}). \tag{4.25}$$

The independence of $\nabla_{*[\kappa[\lambda}\tilde{I}_{\mu]\nu]}$ from k^0 is equivalent to

$$\nabla_{*[\kappa[\lambda}C_{\mu]\nu]} = 0 \tag{4.26}$$

and

$$\nabla_{*[\kappa[\lambda}B_{\mu]\nu]} + 2\delta^0_{[\kappa}\nabla_{*[\lambda}C_{\mu]\nu]} + 2\delta^0_{[\lambda}\nabla_{*[\kappa}C_{\nu]\mu]} \stackrel{*}{=} 0, \tag{4.27}$$

while (4.26) itself is equivalent to the existence of a C_ν such that

$$C_{\mu\nu} = \nabla_{*(\mu}C_{\nu)}. \tag{4.28}$$

Since $C_{\mu\nu}$ is independent of k^0, by first considering μ, $\nu = 1, 2, 3$, we can solve (4.28) for a C_ν ($\nu = 1, 2, 3$) which is independent of k^0. Now it follows from (4.18) and (4.25) that $C_{0\nu} = 0$. Together with (4.28), this shows that we can then choose $C_0 = 0$.

Let us put

$$D_{\mu\nu} = \nabla_{*[\mu}C_{\nu]}. \tag{4.29}$$

Then $D_{0\nu} = 0$ also, and (4.27) can be written as

$$\nabla_{*[\kappa[\lambda}B_{\mu]\nu]} + \delta^0_{[\kappa}\nabla_{*\mu]}D_{\lambda\nu} + \delta^0_{[\lambda}\nabla_{*\nu]}D_{\kappa\mu} \stackrel{*}{=} 0. \tag{4.30}$$

On putting $\mu = 0$ in this, we get

$$\nabla_{*\kappa}(\nabla_{*[\lambda}B_{\nu]0} - D_{\lambda\nu}) \stackrel{*}{=} 0. \tag{4.31}$$

Now the $(k^0)^2$ terms in (4.18) give

$$B_{\nu 0} + k^\mu C_{\mu\nu} \stackrel{*}{=} -\mathrm{i}I_{00\nu}, \tag{4.32}$$

from which

$$\nabla_{*[\lambda}B_{\nu]0} + \tfrac{1}{2}k^\mu \nabla_{*\mu}D_{\lambda\nu} \stackrel{*}{=} 0. \tag{4.33}$$

Since $D_{0\nu} = 0$, and since the integration of (4.28) for C_ν leaves an arbitrary constant in $D_{\mu\nu}$ which is restricted only by $D_{0\nu} = 0$, we see from (4.31) and (4.33) that we may choose C_ν in (4.28) so that

$$k^\mu \nabla_{*\mu}D_{\lambda\nu} + 2D_{\lambda\nu} = 0. \tag{4.34}$$

Consider this along the ray $k^\lambda = ua^\lambda$ through the origin in $T_{z(s)}(M)$, where a^λ is constant and u is a parameter. It gives $u^2 D_{\lambda\nu}$ constant along the ray, and since $D_{\lambda\nu}$ is not singular at the origin, we must have $D_{\lambda\nu} = 0$. Hence from (4.29) there exists a scalar $C(s,k)$ such that

$$C_\lambda = \nabla_{*\lambda}C. \tag{4.35}$$

Since $C_0 = 0$, C is also independent of k^0. The last two terms on the left hand side of (4.30) now vanish, so that there exists a B_λ such that

$$B_{\mu\nu} = \nabla_{*(\mu}B_{\nu)}. \tag{4.36}$$

We again consider first only $\mu, \nu = 1, 2, 3$, and so solve for a $B_\nu (\nu = 1, 2, 3)$ which is independent of k^0. In this case we have from (4.32), (4.28) and (4.35) that

$$B_{0\nu} \stackrel{*}{=} \nabla_{*\nu}(C - k^\mu \nabla_{*\mu}C - \mathrm{i}I_{00\mu}k^\mu). \tag{4.37}$$

We can thus choose

$$B_0 \stackrel{*}{=} 2(C - k^\mu \nabla_{*\mu}C - \mathrm{i}I_{00\mu}k^\mu), \tag{4.38}$$

which is also independent of k^0 by (4.10). We thus finally obtain $\tilde{I}^{\mu\nu}$ in the form

$$\tilde{I}_{\mu\nu}(s,k) \stackrel{*}{=} A_{\mu\nu} + k^0\nabla_{*(\mu}B_{\nu)} + (k^0)^2\nabla_{*\mu\nu}C, \tag{4.39}$$

where $A_{\mu\nu}$, B_ν and C are all independent of k^0.

5. Generating functionals

The next step is to associate a distribution (generalized function) with each of the two generating functions. These will be functionals defined on a space of C^∞ tensorial test functions on TM over τ, of the appropriate type and having compact support. For conciseness, we shall in future take 'test' to imply 'of class C^∞ and compact support'. Then if E_λ and $E_{\lambda\mu} = E_{(\lambda\mu)}$ are such test functions, and their Fourier transforms $\tilde{E}_\lambda$ and $\tilde{E}_{\lambda\mu}$ are defined as in (2.25), we put

$$m[E_\lambda] := (2\pi)^{-4} \int \mathrm{d}s \int \tilde{m}^\lambda(s,k)\, \tilde{E}_\lambda(z(s),k)\, \mathrm{D}k \tag{5.1}$$

and

$$I[E_{\lambda\mu}] := (2\pi)^{-4} \int \mathrm{d}s \int \tilde{I}^{\lambda\mu}(s,k)\, \tilde{E}_{\lambda\mu}(z(s),k)\, \mathrm{D}k. \tag{5.2}$$

To ensure the absolute convergence of these k-space integrations, we assume that for fixed s, $\tilde{m}^\lambda(s,k)$ and $\tilde{I}^{\lambda\mu}(s,k)$ diverge as $k \to \infty$ no faster than a polynomial in k. It was shown in §3 of II that these functionals then uniquely determine the corresponding generating functions.

On using (2.27), we can express (4.17) and (4.18) as relations on these functionals. If Ω and Ω_λ are respectively a scalar and vector test function on TM over τ, and we take

$$E_\lambda = \nabla_{*\lambda}\Omega, \quad E_{\lambda\mu} = \nabla_{*(\lambda}\Omega_{\mu)} \tag{5.3}$$

in (5.1) and (5.2), we obtain

$$m[E_\lambda] = \int \mathrm{d}s\, m^\lambda(s)\, E_\lambda(z(s),0) \tag{5.4}$$

and

$$I[E_{\lambda\mu}] = \int \mathrm{d}s [I^{\lambda\mu}(s)\, E_{\lambda\mu}(z,0) + I^{\kappa\lambda\mu}\nabla_{*\kappa}E_{\lambda\mu}(z,0)]. \tag{5.5}$$

The validity of these for all Ω and Ω_λ is precisely equivalent to (4.17) and (4.18).

The relations (4.23) and (4.39) also have their consequences for the functionals. Consider the k-space integral in (5.1) for a fixed value of s. Then it was shown in §5 of II that (4.23) limits the dependence of this integral on $E_\lambda(z,X)$ to the value of

$$E_\lambda \quad \text{and} \quad \nabla_{*\mu}(X^\lambda E_\lambda) \quad \text{on the surface} \quad n_\lambda(s)\, X^\lambda = 0. \tag{5.6}$$

The same method shows that (4.39) similarly limits the dependence of the corresponding integral in (5.2) to the value of

$$E_{\kappa\lambda}, \quad \nabla_{*\mu}(X^\lambda E_{\kappa\lambda}) \quad \text{and} \quad \nabla_{*\mu\nu}(X^\kappa X^\lambda E_{\kappa\lambda}) \quad \text{on} \quad n_\lambda X^\lambda = 0. \tag{5.7}$$

Let $\tilde{\Sigma}(s)$ denote the hyperplane $n_\lambda(s)\, X^\lambda = 0$ in $T_{z(s)}(M)$, and put $\tilde{\Sigma} := \bigcup_s \tilde{\Sigma}(s) \subset TM$. Then (5.6) and (5.7) show that the support $\tilde{S}$ of either of these functionals lies in $\tilde{\Sigma}$. We now make two further assumptions. The first is that $\tilde{S}$ also lies in the domain N of TM on which δ, defined by (3.1), is a diffeomorphism. Then $S(s) := \operatorname{Exp}(\tilde{S} \cap \tilde{\Sigma}(s))$ is a portion of the hypersurface formed by all geodesics through $z(s)$ orthogonal to $n^\lambda(s)$. The second assumption we make is that the hypersurfaces $S(s) \subset M$ are all disjoint. Then Exp maps $\tilde{S}$ diffeomorphically onto its image

$$S := \bigcup_s S(s) \quad \text{in} \quad M,$$

which will be a world tube containing the base line l.

We can now begin to see the link between the moments and the tensor fields J^α and $T^{\alpha\beta}$ which they are to represent. These too have equivalent functionals, namely

$$\phi_\alpha \to \langle J^\alpha, \phi_\alpha \rangle := \int \mathfrak{J}^\alpha \phi_\alpha\, \mathrm{d}^4x \tag{5.8}$$

and

$$\phi_{\alpha\beta} \to \langle T^{\alpha\beta}, \phi_{\alpha\beta} \rangle := \int \mathfrak{T}^{\alpha\beta} \phi_{\alpha\beta}\, \mathrm{d}^4x, \tag{5.9}$$

where

$$\mathfrak{J}^\alpha := J^\alpha \surd(-g), \quad \mathfrak{T}^{\alpha\beta} := T^{\alpha\beta} \surd(-g), \tag{5.10}$$

and ϕ_α and $\phi_{\alpha\beta} = \phi_{(\alpha\beta)}$ are tensorial test functions on M. The conservation equations (1.31) and (1.32) can then be expressed as

$$\left.\begin{aligned}\langle J^\alpha, \nabla_\alpha \psi\rangle &= 0,\\ \langle T^{\alpha\beta}, \nabla_{(\alpha}\psi_{\beta)}\rangle &= \langle J^\alpha, F_{\beta\alpha}\psi^\beta\rangle,\end{aligned}\right\} \tag{5.11}$$

for all scalar and vector test functions ψ and ψ_α respectively. These are reminiscent of (5.4) and (5.5) in so far as both are special cases applicable when the test functions are derivatives of the appropriate type. The moments will ultimately be defined by expressing the functionals (5.8) and (5.9) in terms of m and I in such a way that verifying the relations (5.11) involves evaluating m and I only for the special cases (5.4) and (5.5). In this way the conservation laws will lead to restrictions only on m^λ, $I^{\lambda\mu}$ and $I^{\kappa\lambda\mu}$. These will give the equation of conservation of total charge and the required equations of motion for the momentum and angular momentum.

To this end we need to relate the E_λ and $E_{\lambda\mu}$ of (5.1) and (5.2) to ordinary tensor fields on M, but we note that this need only be done in the neighbourhood of $\tilde{S} \subset TM$. It is necessary to consider the immediate neighbourhood of $\tilde{S}$, rather than just $\tilde{S}$ itself, as (5.6) and (5.7) show that we also need certain derivatives of E_λ and $E_{\lambda\mu}$ on $\tilde{S}$. Now since $\tilde{S} \subset N$, we already have found in §3 a method of associating bitensors with E_λ and $E_{\lambda\mu}$ around $\tilde{S}$; we can put

$$e_\alpha := \operatorname{Exp}_A E_\lambda \quad \text{and} \quad e_{\alpha\beta} := \operatorname{Exp}_A E_{\lambda\mu} \tag{5.12}$$

in accordance with (3.13) and (3.14). It is not yet clear what to do about the z-dependence of these fields, which have scalar character at z, but it is clearly a step in the right direction. We shall deal later with this problem of the z-dependence; for the moment we shall investigate properties of m and I if we consider them as functionals of e_α and $e_{\alpha\beta}$ instead of as functionals of E_λ and $E_{\lambda\mu}$.

Before proceeding, we should note a possible ambiguity. An obvious alternative to (5.12) is to take

$$e^\alpha := \operatorname{Exp}_A E^\lambda, \quad e^{\alpha\beta} := \operatorname{Exp}_A E^{\lambda\mu}. \tag{5.13}$$

We see from (3.15) that (5.12) and (5.13) are not equivalent, and it is not yet clear which is the best definition to adopt. In order to leave our options open, so as to make the best choice in the light of subsequent developments, we shall for the time being make a more general assumption. Let us put $x = \operatorname{Exp} X$ and take

$$\left.\begin{aligned}e_\alpha(s,x) &:= Z_{1\cdot\alpha}^{\lambda} E_\lambda(z(s), X)\\ \text{and}\qquad e_{\alpha\beta}(s,x) &:= Z_{2\cdot\alpha}^{\lambda} Z_{2\cdot\beta}^{\mu} E_{\lambda\mu}(z(s), X),\end{aligned}\right\} \tag{5.14}$$

leaving the bitensors Z_1 and Z_2 unspecified. From this and (3.15), we see that (5.12) and (5.13) correspond to taking the appropriate Z as $-\sigma^{\lambda}_{\cdot\alpha}$ and $H_{\alpha}^{\cdot\lambda}$ respectively. We shall need to suppose that both bitensors satisfy

$$Z^{\lambda}_{\cdot\alpha}\sigma^\alpha(z,x) = -\sigma^\lambda(z,x). \tag{5.15}$$

This is satisfied by both these choices, and also by the parallel propagator $\bar{g}^{\lambda}_{\cdot\alpha}$, which is another reasonable candidate for $Z^{\lambda}_{\cdot\alpha}$. From (5.14), (5.15), (3.7) and (3.11), we see that the dependence expressed in (5.6) and (5.7) for some fixed s now becomes one on the values of

$$e_\alpha, \quad \nabla_\alpha(\sigma^\beta e_\beta), \quad e_{\alpha\beta}, \quad \nabla_\gamma(\sigma^\beta e_{\alpha\beta}), \quad \nabla_{\gamma\delta}(\sigma^\alpha\sigma^\beta e_{\alpha\beta}) \quad \text{for} \quad x \in S(s). \tag{5.16}$$

As with the notation Exp_A and Exp^A, it is convenient to have a concise notation for relations such as (5.14). If $P^{\lambda\cdots}{}_{\mu\ldots}$ is a tensor field on TM over τ, and if a bitensor $p^{\alpha\cdots}{}_{\beta\ldots}$ satisfies

$$p^{\alpha\cdots}{}_{\beta\ldots}(z,x) = Z_{1\lambda}^{\cdot\alpha} \ldots Z_{1\cdot\beta}^{\mu} \ldots P^{\lambda\cdots}{}_{\mu\ldots}(z,X)$$

in some region $R \subset \delta(N)$, we shall write

$$p = Z_1(P) \quad \text{in} \quad R, \qquad P = Z^1(p) \quad \text{in} \quad \delta^{-1}(R).$$

In (5.14), the bitensors e_α and $e_{\alpha\beta}$ are only defined when their first argument point $z \in l$. They are thus more properly regarded as fields on $l \times M$ rather than on $M \times M$. This distinction is of little significance, but we shall later need to discuss functions defined in 'a neighbourhood of $\delta(\tilde{S})$'. Since $\delta(\tilde{S}) \subset l \times M \subset M \times M$, and the open sets of $l \times M$ are not open in $M \times M$, these neighbourhoods will differ accordingly as $\delta(\tilde{S})$ is considered as a surface in $l \times M$ or in $M \times M$. We shall generally leave this to be deduced from the context, to save a further addition to the notation. It is also convenient to consider $S(s)$ both as a surface in M and as the surface

$$\{(z(s), x);\ x \in S(s)\} \quad \text{of} \quad l \times M,$$

so that we can unambiguously refer to e_α on $S(s)$ without having to additionally specify that the first argument of e_α is also s. With these conventions, $S = \bigcup_s S(s)$ qua subsets of M, while $\delta(\tilde{S}) = \bigcup_s S(s)$ qua subsets of $l \times M$.

We can now make a connexion with ordinary tensor fields on M. Since the hypersurfaces $S(s)$ are disjoint, there is a well defined C^∞ scalar function $\tau(x)$ on S given by

$$\tau(x) = s \quad \text{if} \quad x \in S(s). \tag{5.17}$$

Then the values of e_α and $e_{\alpha\beta}$ on $\delta(\tilde{S})$ are determined by the C^∞ tensor fields

$$\phi_\alpha(x) := e_\alpha(\tau(x), x), \quad \phi_{\alpha\beta}(x) := e_{\alpha\beta}(\tau(x), x) \tag{5.18}$$

on $S \subset M$.

A knowledge of these tensor fields alone is in general not sufficient to determine $m[E_\lambda]$ and $I[E_{\lambda\mu}]$, because of the occurrence of the derivative terms in (5.16). However, there are special classes of fields E_λ and $E_{\lambda\mu}$ for which this *is* enough. The simplest of these is if

$$\sigma^\alpha e_\alpha = 0 \quad \text{and} \quad \sigma^\beta e_{\alpha\beta} = 0 \tag{5.19}$$

in the neighbourhood of $\delta(\tilde{S})$, which makes the derivative terms in (5.16) all vanish. This is assured from the definitions of e_α and $e_{\alpha\beta}$ if

$$X^\lambda E_\lambda = 0 \quad \text{and} \quad X^\mu E_{\lambda\mu} = 0 \tag{5.20}$$

identically. For such E_λ and $E_{\lambda\mu}$, $m[E_\lambda]$ and $I[E_{\lambda\mu}]$ can thus be considered as functionals of the tensor fields ϕ_α and $\phi_{\alpha\beta}$. This is clearly another step towards relating m and I to the functionals (5.8) and (5.9), but since the class of allowed fields E_λ and $E_{\lambda\mu}$ is being restricted, we need to investigate to what extent are the moments determined by m and I when they are restricted to such fields. Now it was shown in §6 of II that $X^\lambda E_\lambda = 0$ if and only if E_λ has the form

$$E_\lambda = X^\mu H_{\mu\lambda}, \quad H_{\mu\lambda} = H_{[\mu\lambda]}, \tag{5.21}$$

for a suitable tensorial test function $H_{\mu\lambda}$ on TM over τ. The same method shows that $X^\mu E_{\lambda\mu} = 0$ if and only if $E_{\lambda\mu}$ has the similar form

$$E_{\kappa\lambda} = X^\mu X^\nu H_{\kappa\mu\lambda\nu}, \quad H_{\kappa\lambda\mu\nu} = H_{[\mu\nu][\kappa\lambda]}, \quad H_{\kappa[\lambda\mu\nu]} = 0. \tag{5.22}$$

These tensors $H_{\dots}$ have the irreducible symmetries [1, 1] and [2, 2] respectively.

From these forms we see that

$$E_\lambda = 0, \quad E_{\lambda\mu} = 0, \quad \nabla_{*\kappa} E_{\lambda\mu} = 0 \quad \text{when} \quad X^\mu = 0. \tag{5.23}$$

Now the contributions to $m[E_\lambda]$ from m^λ, and to $I[E_{\lambda\mu}]$ from $I^{\lambda\mu}$ and $I^{\kappa\lambda\mu}$, are given by the right hand sides of (5.4) and (5.5) for *any* E_λ and $E_{\lambda\mu}$. From (5.23) we thus see that these contributions vanish when (5.20) is satisfied, and hence the restricted forms of m and I discussed above can give no information about these low-order moments. They do, however, enable all the higher moments to be determined completely. To see this, we take the Fourier transforms of (5.21) and (5.22) to give

$$\tilde{E}^{\lambda} = -\mathrm{i}\nabla_{*\mu}\tilde{H}^{\mu\lambda}, \quad \tilde{E}^{\kappa\lambda} = -\nabla_{*\mu\nu}\tilde{H}^{\kappa\mu\lambda\nu}. \tag{5.24}$$

On putting these into (5.1) and (5.2) we obtain

$$m[E_{\lambda}] = (2\pi)^{-4}\mathrm{i}\int \mathrm{d}s \int \nabla_{*[\lambda}\tilde{m}_{\mu]}\tilde{H}^{\lambda\mu}\,\mathrm{D}k \tag{5.25}$$

and

$$I[E_{\lambda\mu}] = -(2\pi)^{-4}\int \mathrm{d}s \int \nabla_{*[\kappa[\mu}\tilde{I}_{\lambda]\,\nu]}\tilde{H}^{\kappa\lambda\mu\nu}\,\mathrm{D}k. \tag{5.26}$$

Let

$$\tilde{Q}_{\lambda\mu}(s,k) := \mathrm{i}\nabla_{*[\lambda}\tilde{m}_{\mu]} \quad \text{and} \quad \tilde{J}_{\kappa\lambda\mu\nu}(s,k) := -\nabla_{*[\kappa[\mu}\tilde{I}_{\lambda]\,\nu]}. \tag{5.27}$$

These have exactly the same symmetries as the H's of (5.21) and (5.22). Since the H's are tensorial test functions which are arbitrary apart from their symmetry requirements, we thus see from (5.25) and (5.26) that $\tilde{Q}_{\lambda\mu}$ and $\tilde{J}_{\kappa\lambda\mu\nu}$ are completely determined by these functionals of the H's From (4.15) and (4.16), we obtain

$$\tilde{Q}_{\lambda\mu} = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}k^{\kappa_1}\dots k^{\kappa_n}Q_{\kappa_1\dots\kappa_n\lambda\mu} \tag{5.28}$$

and

$$\tilde{J}_{\lambda\mu\nu\rho} = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}k^{\kappa_1}\dots k^{\kappa_n}J_{\kappa_1\dots\kappa_n\lambda\mu\nu\rho}, \tag{5.29}$$

where

$$Q_{\kappa_1\dots\kappa_n\lambda\mu} := m_{\kappa_1\dots\kappa_n[\lambda\mu]} \tag{5.30}$$

and

$$J_{\kappa_1\dots\kappa_n\lambda\mu\nu\rho} := I_{\kappa_1\dots\kappa_n[\lambda[\nu\mu]\,\rho]} \tag{5.31}$$

for $n \geq 0$. As a consequence of (4.1), (4.4), (4.7), and (4.11), the Q's and J's satisfy

$$\left.\begin{aligned} Q_{\kappa_1\dots\kappa_n\lambda\mu} &= Q_{(\kappa_1\dots\kappa_n)[\lambda\mu]} \quad \text{for} \quad n \geq 0,\\ Q_{\kappa_1\dots\kappa_{n-1}[\kappa_n\lambda\mu]} &= 0 \quad \text{for} \quad n \geq 1, \end{aligned}\right\} \tag{5.32}$$

and

$$\left.\begin{aligned} J_{\kappa_1\dots\kappa_n\lambda\mu\nu\rho} &= J_{(\kappa_1\dots\kappa_n)[\lambda\mu][\nu\rho]} \quad \text{for} \quad n \geq 0,\\ J_{\kappa_1\dots\kappa_n\lambda[\mu\nu\rho]} = 0 \quad \text{for} \quad n \geq 0, &\quad J_{\kappa_1\dots\kappa_{n-1}[\kappa_n\lambda\mu]\nu\rho} = 0 \quad \text{for} \quad n \geq 1, \end{aligned}\right\} \tag{5.33}$$

and on using these, we can invert (5.30) and (5.31) to give

$$m^{\lambda_1\dots\lambda_n\mu} = [2n/(n+1)]\,Q^{(\lambda_1\dots\lambda_n)\mu} \quad \text{for} \quad n \geq 1, \tag{5.34}$$

and

$$I^{\lambda_1\dots\lambda_n\mu\nu} = [4(n-1)/(n+1)]\,J^{(\lambda_1\dots\lambda_{n-1}|\mu|\lambda_n)\nu} \quad \text{for} \quad n \geq 2. \tag{5.35}$$

Since $\tilde{Q}_{\lambda\mu}$ and $\tilde{J}_{\lambda\mu\nu\rho}$ determine all the $Q^{\cdots}$'s and $J^{\cdots}$'s, we thus see that they determine completely the m's and I's given in (5.34) and (5.35) as required.

The conditions (5.32) and (5.33) are the symmetries corresponding to $[n+1, 1]$ and $[n+2, 2]$ if the symmetrizations of the corresponding Young diagram are performed before the antisymmetrizations. If the antisymmetrizations are performed first, we obtain the properties of the m's and I's. The Q's and J's form an alternative description of the higher moments (i.e. other than m^{λ}, $I^{\lambda\mu}$, $I^{\kappa\lambda\mu}$) which for some purposes is more convenient than that given by the m's and I's. We see, for example, that the orthogonality conditions (4.5) and (4.14) take the simple forms

$$n^{\kappa_1}Q_{\kappa_1\dots\kappa_n\lambda\mu} = 0, \quad n^{\kappa_1}J_{\kappa_1\dots\kappa_n\lambda\mu\nu\rho} = 0 \quad \text{if} \quad n \geq 1. \tag{5.36}$$

We shall use, from now on, whichever is most convenient in a given context.

Let us now summarize the main results of this section so far. By studying the functionals (5.1) and (5.2), we have seen that two special types of test function have particular significance:

(i) If m and I are restricted to act on functions of the form (5.3), then in virtue of the symmetry properties of the moments, these functionals take values depending only on the moments m^λ, $I^{\lambda\mu}$ and $I^{\kappa\lambda\mu}$. These particular moments are conversely determined by these restricted functionals.

(ii) If m and I are restricted to act on functions of the forms (5.21) and (5.22) respectively, they take values independent of the moments mentioned in (i), but they determine all the remaining moments. In virtue of the orthogonality properties of the moments, these restricted functionals may be considered as functionals of the ordinary tensor fields $\phi_\alpha(x)$ and $\phi_{\alpha\beta}(x)$ on M defined as in (5.18). Moreover, in this case the integrands of the s-integrals in (5.1) and (5.2) depend, for fixed s, only on the restrictions of ϕ_α and $\phi_{\alpha\beta}$ to the hypersurface $S(s)$ defined in the paragraph following equation (5.7).

There is a third special case which is also of importance. In selecting (5.19), we made the derivative contributions to (5.16) vanish. If this is not to be so, but we still wish to have m and I determined by the tensor fields ϕ_α and $\phi_{\alpha\beta}$ of (5.18), then these derivatives must be determined by ϕ_α and $\phi_{\alpha\beta}$. Now differentiation of (5.18) yields

$$\nabla_\alpha \phi_\beta = \nabla_\alpha e_\beta + (\partial e_\beta/\partial s)\, \partial_\alpha \tau. \tag{5.37}$$

On using this and similar relations for $e_{\alpha\beta}$, we may deduce from (5.1), (5.2) and (5.16) that

$$\left.\begin{aligned} m[E_\lambda] &\text{ depends on } e_\alpha \text{ and } \sigma^\alpha \partial e_\alpha/\partial s \text{ on } \delta(\tilde{S}), \\ I[E_{\lambda\mu}] &\text{ depends on } e_{\alpha\beta},\ \sigma^\beta \partial e_{\alpha\beta}/\partial s \text{ and } \sigma^\alpha\sigma^\beta \nabla_\gamma(\partial e_{\alpha\beta}/\partial s) \text{ on } \delta(\tilde{S}). \end{aligned}\right\} \tag{5.38}$$

So m and I are determined by ϕ_α and $\phi_{\alpha\beta}$ if

$$\sigma^\alpha \partial e_\alpha/\partial s = 0, \quad \sigma^\beta \partial e_{\alpha\beta}/\partial s = 0, \quad \sigma^\alpha\sigma^\beta \nabla_\gamma(\partial e_{\alpha\beta}/\partial s) = 0 \quad \text{on} \quad \delta(\tilde{S}). \tag{5.39}$$

This has the advantage over case (ii) above in that, given any ϕ_α and $\phi_{\alpha\beta}$, an e_α and $e_{\alpha\beta}$ can be found satisfying (5.18) and (5.39) merely by requiring (5.18) on $\delta(\tilde{S})$ and putting $\partial e_\alpha/\partial s = 0$ and $\partial e_{\alpha\beta}/\partial s = 0$ in the neighbourhood of $\delta(\tilde{S})$. In contrast, case (ii) above needs $\sigma^\alpha \phi_\alpha = 0$ and $\sigma^\alpha \phi_{\alpha\beta} = 0$ on $\delta(\tilde{S})$.

The price we pay for this advantage is that the s-integrands in (5.1) and (5.2) no longer depend on the restrictions of ϕ_α and $\phi_{\alpha\beta}$ to a particular $S(s)$. The complications caused by this are well illustrated by the treatment of J^α in II, and they are considerably greater for $T^{\alpha\beta}$. Guided by the work of §5 of II, we now find a correspondence between fields e_α and $e_{\alpha\beta}$ satisfying (5.39), and those satisfying (5.19).

6. The auxiliary functions ω and ω_α

With a view towards later substitution into the functionals (5.8) and (5.9), which satisfy the relations (5.11), we shall consider correspondences of the form

$$e_\alpha \to c_\alpha := e_\alpha + \partial_\alpha \omega, \tag{6.1}$$

$$e_{\alpha\beta} \to c_{\alpha\beta} := e_{\alpha\beta} + \nabla_{(\alpha}\omega_{\beta)}, \tag{6.2}$$

where all the fields have scalar character at $z(s)$ as well as the tensor character at x indicated by their indices. Due to this s-dependence, we cannot directly apply (5.11) to the contributions from

ω and ω_α, but this will not cause difficulty as we shall use relations of the form (5.37). Our immediate aim is to obtain correspondences (6.1) and (6.2) in which all the fields have compact support, and which satisfy

$$\sigma^\alpha e_\alpha = 0 \tag{6.3}$$

and

$$\sigma^\alpha e_{\alpha\beta} = 0 \tag{6.4}$$

identically, and

$$\sigma^\alpha \partial c_\alpha/\partial s = 0 \tag{6.5}$$

and

$$\sigma^\beta \partial c_{\alpha\beta}/\partial s, \quad \sigma^\alpha\sigma^\beta\nabla_\gamma(\partial c_{\alpha\beta}/\partial s) = 0 \tag{6.6}$$

for each s and all x in some neighbourhood of $S(s)$. These are the two types of special case considered in the previous section, as given by (5.19) and (5.39). We shall need to work in both directions, i.e. given e_α and $e_{\alpha\beta}$, find all possible c_α and $c_{\alpha\beta}$, and vice versa, but when c_α and $c_{\alpha\beta}$ are given, we shall feel free to alter them outside of some neighbourhood of $S(s)$ to ensure that an appropriate e_α or $e_{\alpha\beta}$ of compact support does exist. In view of our results on the support of the functionals m and I, we do not expect this indeterminancy to cause any difficulties later on.

Consider first (6.1). Initially, let $e_\alpha(s, x)$ be given, with compact support but not necessarily satisfying (6.3). To find c_α satisfying (6.5), substitute from (6.1) into (6.5) to give

$$\sigma^\alpha \partial_\alpha(\partial\omega/\partial s) = -\sigma^\alpha \partial e_\alpha/\partial s \tag{6.7}$$

in some neighbourhood of $S(s)$. If $x(u)$ is any geodesic through $z(s)$, with $x(0) = z(s)$ and u as an affine parameter, then (6.7) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial\omega}{\partial s}\right) = -\dot{x}^\alpha \frac{\partial}{\partial s} e_\alpha, \tag{6.8}$$

where $\dot{x}^\alpha := \mathrm{d}x^\alpha/\mathrm{d}u$, holding along all such geodesics in the required neighbourhood. This can be integrated for arbitrary initial values of $\partial\omega/\partial s$ at $u = 0$, i.e. at $x = z(s)$. Having thus obtained $\partial\omega/\partial s$ in a neighbourhood of $S(s)$, we can obtain ω by integrating with respect to s using, for each x, an arbitrary initial value at $s = \tau(x)$. This shows that we can specify

$$[\partial\omega/\partial s]_{x=z(s)} = a(s), \quad \omega(\tau(x), x) = \nu(x) \tag{6.9}$$

subject only to $a(s)$ and $\nu(x)$ being C^∞ and of compact support. Since c_α is required to satisfy (6.5) only in a neighbourhood of $S(s)$, we simply choose $\omega(s, x)$ to be any biscalar of compact support which agrees with this construction in this neighbourhood, and then take (6.1) as defining the corresponding c_α.

When (6.3) is satisfied, it gives by differentiation with respect to s that

$$\sigma^\alpha \partial e_\alpha/\partial s = -v^\lambda \sigma^\alpha{}_{;\lambda} e_\alpha, \tag{6.10}$$

where $v^\lambda := \mathrm{d}z^\lambda/\mathrm{d}s$. In this case, (6.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial\omega}{\partial s}\right) = u^{-1} v^\lambda \sigma_\lambda{}^\alpha e_\alpha. \tag{6.11}$$

The singularity on the right hand side at $u = 0$ is here only apparent, since (6.3) implies that

$$e_\lambda(s, z(s)) = 0, \tag{6.12}$$

and hence $e_\alpha(s, x(u)) = O(u)$ as $u \to 0$.

Now, conversely, suppose that $c_\alpha(s,x)$ is given. Again, we do not initially assume that (6.5) is satisfied. To find an e_α satisfying (6.3), substitute from (6.1) into (6.3) to give

$$\mathrm{d}\omega/\mathrm{d}u = \dot{x}^\alpha c_\alpha \tag{6.13}$$

along the geodesic segments $x(u)$ as above. This may be integrated throughout a neighbourhood of $S(s)$, with an arbitrary initial value at $u = 0$, say

$$\omega(s, z(s)) = A(s). \tag{6.14}$$

The corresponding e_α, given by (6.1), will then satisfy (6.3) in this neighbourhood. Now extend e_α and ω outside this neighbourhood in an arbitrary manner, subject only to (6.3), so that they have compact support, and use (6.1) now to define a new c_α. In accordance with our comments above, this c_α will agree with the original one in some neighbourhood of $S(s)$, for each s.

If we start with an arbitrary e_α satisfying (6.3) and construct c_α using the initial values (6.9), then we may recover ω from c_α using (6.13) provided that we take the initial value (6.14) to be compatible with (6.9). This requires taking

$$A(s) = \nu(z(s)). \tag{6.15}$$

Conversely, if we start from a c_α satisfying (6.5) and find e_α using the initial value (6.14), then to recover ω from e_α we must use the appropriate values of $\nu(x)$ and $a(s)$. These are not determined by e_α and $A(s)$ alone, but are determined if we additionally know

$$\phi_\alpha(x) := c_\alpha(\tau(x), x), \tag{6.16}$$

which is the function introduced in equation (5.18). For then (6.13) and (6.9) give

$$\mathrm{d}\nu/\mathrm{d}u = \dot{x}^\alpha \phi_\alpha \tag{6.17}$$

for those geodesics $x(u)$ in $S(s)$, which when integrated with the initial condition (6.15), determines ν on each $S(s)$, and hence throughout S. To find $a(s)$, we then use (6.12), (6.1) and (6.16) to give $\partial_\kappa \omega = \phi_\kappa$ at $x = z(s)$, and hence with (6.9) and (6.14) we get

$$a = \mathrm{d}A/\mathrm{d}s - v^\kappa \phi_\kappa. \tag{6.18}$$

Given c_α satisfying (6.5), another choice for c_α which yields the same $\phi_\alpha(x)$ is $\hat{c}_\alpha(s,x) := \phi_\alpha(x)$. Equation (6.5) is trivial in this case, as $\hat{c}_\alpha$ is independent of s. Since e_α, A and ϕ_α together determine c_α in the neighbourhood of $S(s)$, it follows that e_α must change if we alter c_α while keeping A and ϕ_α fixed. Hence if we construct $\hat{e}_\alpha$ and $\hat{c}_\alpha$ using the same value of A as before, then $e_\alpha \neq \hat{e}_\alpha$. It will be important for us to know that nevertheless

$$e_\alpha(s,x) = \hat{e}_\alpha(s,x) \quad \text{for all} \quad x \in S(s). \tag{6.19}$$

Clearly (6.17) shows that $\hat{\nu} = \nu$, and hence that

$$\omega = \hat{\omega} \quad \text{on} \quad S(s), \tag{6.20}$$

but from (6.1) we see that (6.19) requires that also $\partial_\alpha(\omega - \hat{\omega}) = 0$ on $S(s)$. We now prove that this is so. We first note that the techniques used in the proof of (5.38) show that, because of (6.20), $\partial_\alpha \omega$ and $\partial_\alpha \hat{\omega}$ will agree on $S(s)$ if and only if $\partial\omega/\partial s$ and $\partial\hat{\omega}/\partial s$ agree there. Let us put $\overline{\omega} = \omega - \hat{\omega}$, and similarly define $\overline{\nu}(x)$, $\overline{a}(s)$, $\overline{c}_\alpha$ and $\overline{e}_\alpha$. Then from (6.1) and (6.9) we have, on $S(s)$,

$$\overline{c}_\alpha = \overline{e}_\alpha + \partial_\alpha \overline{\nu} - (\partial\overline{\omega}/\partial s)\, \partial_\alpha \tau. \tag{6.21}$$

But we have seen that $\overline{\nu}(x) = 0$, and from the definition of $\hat{c}_\alpha$ we have that $\overline{c}_\alpha(s,x) = 0$ for $x \in S(s)$.

Hence (6.21) gives

$$\bar{e}_\alpha = (\partial\bar{\varpi}/\partial s)\,\partial_\alpha\tau \quad \text{on} \quad S(s). \tag{6.22}$$

Since (6.12) implies that $\bar{e}_\alpha(s, z(s)) = 0$, (6.22) then shows that

$$\partial\bar{\varpi}/\partial s = 0 \quad \text{when} \quad x = z(s), \tag{6.23}$$

i.e. $\bar{a}(s) = 0$. Now on substituting (6.22) into (6.11), we see that

$$u\frac{\mathrm{d}}{\mathrm{d}u}\ln\frac{\partial\bar{\varpi}}{\partial s} = v^\lambda\sigma_{;\lambda}^{\;\;\alpha}\partial_\alpha\tau \tag{6.24}$$

along those geodesics $x(u)$ in $S(s)$. As the coincidence limit $\langle\sigma_{;\lambda}^{\;\;\alpha}\rangle = -\delta_\lambda^\alpha$ and $v^\lambda\partial_\lambda\tau = 1$, we see that the right hand side of (6.24) is $-1+O(u)$. The general solution of (6.24) thus has the form

$$\partial\bar{\varpi}/\partial s = ku^{-1}\exp f(u), \tag{6.25}$$

where k is an arbitrary constant and $f(u)$ is differentiable in a neighbourhood of $u = 0$. The only solution compatible with (6.23) is then when $k = 0$, and so $\partial\bar{\varpi}/\partial s = 0$ on $S(s)$, as required, which completes the proof.

For given e_α, we now separate out in the corresponding ϕ_α the dependence on $\nu(x)$ and $a(s)$. From (6.16), (6.1) and (6.9) we see that

$$\phi_\alpha(x) = e_\alpha(s, x) + \partial_\alpha\nu - (\partial\omega/\partial s)\,\partial_\alpha\tau \tag{6.26}$$

if $x \in S(s)$. But from (6.8) and (6.9) we see that

$$\partial\omega/\partial s = {}'\psi(s, x) + a(s) \tag{6.27}$$

if ${}'\psi(s, x)$ is the solution for $\partial\omega/\partial s$ corresponding to $a(s) = 0$. If we now put

$$\psi(x) := {}'\psi(\tau(x), x), \tag{6.28}$$

then (6.26) gives

$$\phi_\alpha(x) = e_\alpha(\tau(x), x) + \partial_\alpha\nu(x) - (\psi(x) + a(\tau(x)))\,\partial_\alpha\tau, \tag{6.29}$$

as required.

We turn now to the corresponding treatment of (6.2). Consider first the conditions (6.6) on $c_{\alpha\beta}$. By applying the operator $\sigma^\gamma(z(s), x)\,\nabla_\gamma$ to the first of these, we find that it implies also

$$\sigma^\alpha\sigma^\beta\nabla_\alpha\,\partial c_{\beta\gamma}/\partial s = 0. \tag{6.30}$$

Note for later use that since $\sigma^\gamma\nabla_\gamma$ differentiates along geodesics through $z(s)$, this implication holds also if, as in (5.39),

$$\sigma^\alpha\,\partial c_{\alpha\beta}/\partial s = 0 \tag{6.31}$$

is required to hold only *on* $S(s)$, rather than in a neighbourhood of $S(s)$ as is the case in (6.6). On using (6.30), we can replace (6.6) by the equivalent pair of equations (6.31) and

$$\sigma^\alpha\sigma^\beta\nabla_{\{\alpha}\frac{\partial}{\partial s}c_{\gamma\beta\}} = 0, \tag{6.32}$$

where the $\{\,\}$ notation is defined in appendix 1.

The next step depends essentially on the fact that we are presently requiring (6.31) and (6.32) to hold in the neighbourhood of $S(s)$. For this enables us to apply ∇_γ itself to (6.31). We then easily deduce that (6.31) actually implies (6.32). Equation (6.31) also implies that

$$\partial c_{\kappa\lambda}/\partial s = 0 \quad \text{when} \quad x = z(s). \tag{6.33}$$

We now show conversely that (6.32), together with the initial value (6.33), implies (6.31), so that the two conditions (6.31), and (6.32) with (6.33), are equivalent. To see this, let $x(u)$ be a geodesic through $z(s)$ as in (6.8). On multiplying (6.32) by σ^γ and using $\sigma^\gamma = u\dot{x}^\gamma$, we get

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{x}^\alpha \dot{x}^\beta \frac{\partial}{\partial s} c_{\alpha\beta}\right) = 0. \tag{6.34}$$

Because of the initial condition (6.33), this implies

$$\sigma^\alpha \sigma^\beta \partial c_{\alpha\beta}/\partial s = 0 \tag{6.35}$$

in a neighbourhood of $S(s)$. From (6.32) and (6.35) we now get that along all geodesics $x(u)$ as above,

$$u\frac{\delta}{\mathrm{d}u}\left(\sigma^\alpha \frac{\partial}{\partial s} c_{\alpha\gamma}\right) + \sigma^\alpha(\sigma^\beta_{\cdot\gamma} - A^\beta_\gamma)\frac{\partial}{\partial s} c_{\alpha\beta} = 0. \tag{6.36}$$

On using (3.10), this can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\sigma^\alpha H^\beta_{\cdot\kappa} \frac{\partial}{\partial s} c_{\alpha\beta}\right) = 0, \tag{6.37}$$

which can be integrated to give (6.31) as required, on again using (6.33) as initial condition.

For the moment, (6.32) with (6.33) will be the most useful form of the conditions on $c_{\alpha\beta}$. Using it, let us follow for (6.2) the development of (6.1) that starts with (6.7). So let $e_{\alpha\beta}(s, x)$ be given, with compact support but not necessarily satisfying (6.4). If we substitute from (6.2) into (6.32) and let $x^\alpha(u)$ be a geodesic as in (6.8), we obtain

$$\frac{\delta^2}{\mathrm{d}u^2}\left(\frac{\partial}{\partial s}\omega_\alpha\right) + R_{\alpha\beta\gamma\delta}\dot{x}^\beta \dot{x}^\gamma \frac{\partial}{\partial s}\omega^\delta = -\dot{x}^\beta \dot{x}^\gamma \nabla_{\{\beta}\frac{\partial}{\partial s} e_{\alpha\gamma\}}. \tag{6.38}$$

By integration of this along all such geodesics in a neighbourhood of $S(s)$, we obtain a $\partial\omega_\alpha/\partial s$ which ensures that (6.32) will be satisfied. As before, we can then find a corresponding ω_α satisfying

$$\omega_\alpha(\tau(x), x) = \nu_\alpha(x) \tag{6.39}$$

for arbitrary ν_α of compact support. To specify uniquely a solution of (6.38), we need to give $\partial\omega_\alpha/\partial s$ and $\nabla_\beta(\partial\omega_\alpha/\partial s)$ at $u = 0$. In part, these are determined by (6.33), which from (6.2) is equivalent to

$$\nabla_{(\kappa}\frac{\partial}{\partial s}\omega_{\lambda)} = -\frac{\partial}{\partial s} e_{\kappa\lambda} \tag{6.40}$$

where, for example, $\nabla_\kappa \omega_\lambda$ should be taken as meaning $\delta^{\alpha\beta}_{\kappa\lambda} \lim\limits_{x\to z} \nabla_\alpha \omega_\beta$. Using this same convention, we are thus free to specify

$$\frac{\partial}{\partial s}\omega_\kappa = a_\kappa(s) \quad \text{and} \quad \nabla_{[\kappa}\frac{\partial}{\partial s}\omega_{\lambda]} = b_{\kappa\lambda}(s) \tag{6.41}$$

arbitrarily, subject only to compact support and $b_{(\kappa\lambda)} = 0$.

To separate out from $\partial\omega_\alpha/\partial s$ its dependence on a_κ and $b_{\kappa\lambda}$, we observe that if the right hand side of (6.38) were zero, it would be the equation of geodesic deviation. The results of §3 of I then give the general solution of (6.38) as

$$\partial\omega_\alpha/\partial s = K_{\alpha}^{\cdot\kappa} a_\kappa + H_{\alpha}^{\cdot\kappa}\sigma^\lambda b_{\kappa\lambda} + {}'\psi_\alpha, \tag{6.42}$$

where ${}'\psi_\alpha(s,x)$ is the solution for $\partial\omega_\alpha/\partial s$ corresponding to $a_\kappa = 0$, $b_{\kappa\lambda} = 0$. If

$$\phi_{\alpha\beta}(x) := c_{\alpha\beta}(\tau(x), x) \tag{6.43}$$

as in (5.18), then from (6.39) and (6.2) we get, as the analogue of (6.26), that

$$\phi_{\alpha\beta} = e_{\alpha\beta} + \nabla_{(\alpha}v_{\beta)} - (\partial\omega_{(\alpha}/\partial s)\,\partial_{\beta)}\tau, \tag{6.44}$$

where $s = \tau(x)$. The dependence of $\phi_{\alpha\beta}$ on a_κ and $b_{\kappa\lambda}$ is obtained by substituting (6.42) into (6.44). By analogy with (6.28), we also write

$$\psi_\alpha(x) := {}'\psi_\alpha(\tau(x), x). \tag{6.45}$$

We now see what simplifications occur when $e_{\alpha\beta}$ satisfies (6.4). First, note that this implies

$$\sigma^\alpha\sigma^\beta\partial e_{\alpha\beta}/\partial s = 0, \quad \sigma^\alpha\sigma^\beta\nabla_\gamma e_{\alpha\beta} = 0 \quad \text{and} \quad \sigma^\gamma\sigma^\alpha\nabla_\gamma e_{\alpha\beta} = 0. \tag{6.46}$$

By letting $x \to z$ in (6.4) and the first two of these, we can deduce (cf. (5.23)) that

$$e_{\lambda\mu} = 0, \quad \partial e_{\lambda\mu}/\partial s = 0, \quad \nabla_\kappa e_{\lambda\mu} = 0, \tag{6.47}$$

in the notation of (6.40). Hence (6.40) and (6.41) reduce to

$$\partial\omega_\kappa/\partial s = a_\kappa(s), \quad \nabla_\kappa\partial\omega_\lambda/\partial s = b_{\kappa\lambda}(s). \tag{6.48}$$

We next show that when $a_\kappa = 0$, which is the case in the construction of ${}'\psi_\alpha$, one integration of (6.38) can be performed explicitly. Although this can be done directly, it is easier to derive the first order equation separately, from the alternative condition (6.31) on c_α. We first deduce from (6.31), (6.2) and the first of equations (6.46) that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{x}^\alpha\frac{\partial}{\partial s}\omega_\alpha\right) = 0 \tag{6.49}$$

along the usual geodesic segments $x(u)$, with $x(0) = z(s)$, in some neighbourhood of $S(s)$. Since $a_\kappa = 0$, this integrates to show that

$$\sigma^\alpha\partial\omega_\alpha/\partial s = 0 \tag{6.50}$$

in this neighbourhood. If we now substitute from (6.2) into (6.31) and use (6.50) and (6.4), we find that

$$u\frac{\delta}{\mathrm{d}u}\left(\frac{\partial}{\partial s}\omega_\alpha\right) - \sigma_\alpha{}^\beta\frac{\partial}{\partial s}\omega_\beta = 2v^\lambda\sigma_\lambda{}^\beta e_{\alpha\beta} \tag{6.51}$$

along such geodesic segments, which can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{-1}\sigma^{\kappa\alpha}\frac{\partial}{\partial s}\omega_\alpha\right) = 2u^{-2}v_\lambda\sigma^{\kappa\alpha}\sigma^{\lambda\beta}e_{\alpha\beta}. \tag{6.52}$$

As with (6.11), the singularity on the right hand side is only apparent, as (6.47) shows that $e_{\alpha\beta}(s, x(u)) = O(u^2)$ as $u \to 0$. All solutions of (6.52) satisfy $\partial\omega_\kappa/\partial s = 0$ at $u = 0$; the boundary condition at $u = 0$ is the freedom to specify

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial}{\partial s}\omega_\kappa\right) = \dot{x}^\lambda b_{\lambda\kappa} \quad \text{by} \quad (6.48).$$

We next assume $c_{\alpha\beta}$ given, although not necessarily satisfying (6.6), and we seek an ω_α such that $e_{\alpha\beta}$ satisfies (6.4). The remarks following (6.14) concerning modifying c_α are equally applicable in this case to $c_{\alpha\beta}$, and so we shall only concern ourselves with the corresponding equations.

First note that (6.46) implies

$$\sigma^\beta \sigma^\gamma \nabla_{\{\beta} e_{\alpha\gamma\}} = 0. \tag{6.53}$$

On substituting from (6.2) into this, we get

$$\delta^2 \omega_\alpha / \mathrm{d}u^2 + R_{\alpha\beta\gamma\delta}\, \dot{x}^\beta \dot{x}^\gamma \omega^\delta = \dot{x}^\beta \dot{x}^\gamma \nabla_{\{\beta} c_{\alpha\gamma\}} \tag{6.54}$$

along the usual geodesic segments. On integrating this with arbitrary boundary conditions at $u = 0$, we get an ω_α which ensures that $e_{\alpha\beta} := c_{\alpha\beta} - \nabla_{(\alpha}\omega_{\beta)}$ satisfies (6.53). However, just as (6.32) and (6.33) together imply (6.31), so also (6.53) implies (6.4) provided that $e_{\kappa\lambda} = 0$ at $u = 0$. The allowed freedom in the initial conditions for (6.54) is thus that we can take

$$\omega_\kappa = A_\kappa(s), \quad \nabla_\kappa \omega_\lambda = c_{\kappa\lambda} + B_{\kappa\lambda}(s) \quad \text{at} \quad u = 0, \tag{6.55}$$

where A_κ and $B_{\kappa\lambda} = B_{[\kappa\lambda]}$ are arbitrary.

If we start with an $e_{\alpha\beta}$ satisfying (6.4), consistency of the two sets of initial conditions for the closed cycle from $e_{\alpha\beta}$ to $c_{\alpha\beta}$ and thence back to $e_{\alpha\beta}$ requires, from (6.55), (6.39) and (6.48) that

$$A_\kappa(s) = \nu_\kappa(z(s)) \quad \text{and} \quad B_{\kappa\lambda} = \nabla_{[\kappa}\nu_{\lambda]} - a_{[\lambda}\partial_{\kappa]}\tau. \tag{6.56}$$

The definition of τ shows that $\partial_\kappa \tau = n_\kappa/(n^\lambda v_\lambda)$, so that the last term in (6.56) is easily evaluated. If, conversely, we start from a $c_{\alpha\beta}$ satisfying (6.6) and find $e_{\alpha\beta}$ using (6.55) as initial conditions, then to recover $c_{\alpha\beta}$ from $e_{\alpha\beta}$, A_κ and $B_{\kappa\lambda}$, we need to be able to reconstruct a_κ, $b_{\kappa\lambda}$ and ν_α. The corresponding treatment of c_α following (6.16) suggests that to do this, we need also to know $\phi_{\alpha\beta}$ as given by (6.43). However, it is not evident that even this suffices, as (6.54), which corresponds to (6.13), involves $\nabla_\alpha c_{\beta\gamma}$. Although $c_{\alpha\beta} = \phi_{\alpha\beta}$ on $S(s)$ by definition, we will not generally have

$$\nabla_\alpha c_{\beta\gamma} = \nabla_\alpha \phi_{\beta\gamma} \quad \text{on} \quad S(s).$$

We now show that, nevertheless, we do have

$$\sigma^\beta \sigma^\gamma \nabla_{\{\beta} c_{\alpha\gamma\}} = \sigma^\beta \sigma^\gamma \nabla_{\{\beta} \phi_{\alpha\gamma\}} \quad \text{on} \quad S(s), \tag{6.57}$$

and so we may replace $c_{\alpha\beta}$ by $\phi_{\alpha\beta}$ on the right hand side of (6.54) when the geodesic $x(u)$ lies in $S(s)$. This may then be integrated on using (6.55) to give ω_α on $S(s)$, and hence ν_α by (6.39). To prove (6.57), let us put $\hat{c}_{\alpha\beta}(s, x) = \phi_{\alpha\beta}(x)$ and $\bar{c}_{\alpha\beta} = c_{\alpha\beta} - \hat{c}_{\alpha\beta}$, as was done for c_α. Then (6.43) implies

$$\nabla_{\{\beta}\bar{c}_{\alpha\gamma\}} = -\partial_{\{\beta}\tau\, \partial c_{\alpha\gamma\}}/\partial s \quad \text{on} \quad S(s), \tag{6.58}$$

from which (6.57) follows on using (6.6).

We also need to express a_κ and $b_{\kappa\lambda}$ in terms of A_κ, $B_{\kappa\lambda}$ and $\phi_{\alpha\beta}$. Using (6.43), (6.48) and (6.55), we see, in the notation of (6.40), that

$$a_\lambda = \delta A_\lambda/\mathrm{d}s - v^\mu(\phi_{\mu\lambda} + B_{\mu\lambda}) \tag{6.59}$$

and

$$b_{\kappa\lambda} = \delta B_{\kappa\lambda}/\mathrm{d}s - v^\mu \nabla_{\mu[\kappa}\omega_{\lambda]}. \tag{6.60}$$

Together with (6.2), (6.47) and (6.55), (6.60) gives

$$b_{\kappa\lambda} = \delta B_{\kappa\lambda}/\mathrm{d}s - v^\mu(2\nabla_{[\kappa} c_{\lambda]\mu} + R_{\kappa\lambda\mu\nu}A^\nu). \tag{6.61}$$

But similarly to the derivation of (6.57), using (6.33) we see that $\nabla_\kappa c_{\lambda\mu} = \nabla_\kappa \phi_{\lambda\mu}$, and hence (6.61) finally becomes

$$b_{\kappa\lambda} = \delta B_{\kappa\lambda}/\mathrm{d}s - v^\mu(2\nabla_{[\kappa}\phi_{\lambda]\mu} + R_{\kappa\lambda\mu\nu}A^\nu). \tag{6.62}$$

We shall also need to know that if $e_{\alpha\beta}$ and $\hat{e}_{\alpha\beta}$ are derived from $c_{\alpha\beta}$ and $\hat{c}_{\alpha\beta}$ using the same initial conditions, then they agree on each $S(s)$. In fact, we show that

$$\omega_\alpha = \hat{\omega}_\alpha, \quad \nabla_\alpha \omega_\beta = \nabla_\alpha \hat{\omega}_\beta \quad \text{and} \quad e_{\alpha\beta} = \hat{e}_{\alpha\beta} \quad \text{on each} \quad S(s). \tag{6.63}$$

We define $\bar{e}_{\alpha\beta}$, etc., as before. The first of the results (6.63) has already been proved. By it, we see that the second is equivalent to

$$\partial \bar{\omega}_\alpha / \partial s = 0 \quad \text{on} \quad S(s), \tag{6.64}$$

while by (6.2), the third follows from the second. It thus remains to prove (6.64). The method is similar to that leading to (6.25). It follows from (6.44), since $\bar{\nu}_\alpha = 0$, that

$$\bar{e}_{\alpha\beta} = \partial_{(\alpha} \tau \frac{\partial}{\partial s} \bar{\omega}_{\beta)} \quad \text{on} \quad S(s). \tag{6.65}$$

As $\bar{e}_{\kappa\lambda} = 0$ by (6.47), this shows that $\partial \bar{\omega}_\kappa / \partial s = 0$, so that by (6.48), $\bar{a}_\kappa(s) = 0$. Hence (6.52) is valid for the barred variables, and on $S(s)$ it gives, with (6.65),

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{-1} \sigma^{\kappa\alpha} \frac{\partial}{\partial s} \bar{\omega}_\alpha\right) = 2u^{-2} v_\lambda \sigma^{\kappa\alpha} \sigma^{\lambda\beta} \partial_{(\alpha} \tau \frac{\partial}{\partial s} \bar{\omega}_{\beta)}. \tag{6.66}$$

We integrate this in two stages. If we first multiply it by v_κ, we get a differential equation in the variable $v_\kappa \sigma^{\kappa\alpha} \partial \bar{\omega}_\alpha / \partial s$, which may be integrated as before to give a solution of the same form as (6.25). This is compatible with $\bar{a}_\kappa = 0$ only if

$$v_\kappa \sigma^{\kappa\alpha} \partial \bar{\omega}_\alpha / \partial s = 0 \quad \text{on} \quad S(s). \tag{6.67}$$

On putting this back into the right hand side of (6.66), we can integrate the resulting equation to give

$$\sigma^{\kappa\alpha} \partial \bar{\omega}_\alpha / \partial s = C^\kappa \exp f(u) \quad \text{on} \quad S(s), \tag{6.68}$$

where C^κ is an arbitrary constant vector at z, and $f(u)$ is differentiable in a neighbourhood of $u = 0$. Again, this is compatible with $\bar{a}_\kappa = 0$ only if $C^\kappa = 0$, from which (6.64) follows as required.

For later convenience we shall write

$$\omega_\alpha(s, x) = \xi_\alpha(s, x) + \lambda_\alpha(s, x), \tag{6.69}$$

where λ_α is the solution of (6.54) corresponding to $A_\kappa = 0$, $B_{\kappa\lambda} = 0$, and ξ_α is the corresponding solution of the homogeneous equation. As this is again the geodesic deviation equation, we have

$$\xi^\alpha = K^\alpha_{.\kappa} A^\kappa + H^\alpha_{.\kappa} \sigma_\lambda B^{\kappa\lambda}, \tag{6.70}$$

as in (6.42).

7. The reduced moments of J^α

We are now in a position to relate the generating functionals of § 5 to the sources J^α and $T^{\alpha\beta}$. In doing so, we shall reverse the order of development used in II, by proving first the existence and then the uniqueness of the reduced moments. This will be done by first making a provisional definition of the moments, from which we deduce both their existence and the result which we then adopt as a new and final definition. The existence of the moments given by the final definition is thus assured, and their uniqueness is then proved.

In the present section we follow this procedure for J^α, being guided by the expected final results which are already known from II. This is to provide a model which we can then follow in the subsequent sections in treating $T^{\alpha\beta}$. Only in this way do we seem able to arrive at the best definition of the moments of $T^{\alpha\beta}$, as it is not a straightforward generalization of that given in II for J^{α}.

First, we note some weak geometrical restrictions that we shall need to impose. These were discussed in I and II, and require, roughly, that the body be not too large, and be convex. The convexity condition is not essential to the method, but it saves us from having to make minor distinctions between the supports of various functionals, and so saves yet more complication in the notation. Similarly, it is convenient to take J^{α} and $T^{\alpha\beta}$ as having the same support, W say.

Assume that the intersection of W with an arbitrary spacelike hypersurface Σ lies in some convex neighbourhood in M whose closure is compact. Choose the base line l for the moments so that its point of intersection with any such Σ lies in such a convex neighbourhood. If $\Sigma(s)$ denotes the hypersurface formed by all geodesics through $z(s)$ orthogonal to $n^{\lambda}(s)$, we further suppose n^{λ} chosen so that the cross-sections $\Sigma(s) \cap W$ of W are spacelike and disjoint. Finally, the 'convexity condition' we use is that the geodesic segment joining $z(s)$ to any point of $\Sigma(s) \cap W$ should lie entirely within this cross-section. These assumptions ensure the validity of the geometrical constructions we shall make.

We also need, temporarily, an assumption connecting W with the support $\tilde{S}$ of $m[E_{\lambda}]$. In the notation of §5, $S(s) \subset \Sigma(s)$. We suppose that also $\Sigma(s) \cap W \subset S(s)$, so that $W \subset S$.

Let $w^{\alpha}(s)$ be any C^{∞} vector field on M such that $w^{\alpha}\,\mathrm{d}s$ drags $\Sigma(s)$ into $\Sigma(s+\mathrm{d}s)$. Then if ϕ_{α} is any vectorial test function on M, we have from (5.8) that

$$\langle J^{\alpha}, \phi_{\alpha}\rangle = \int \mathrm{d}s \int_{\Sigma(s)} \mathfrak{J}^{\alpha}\phi_{\alpha} w^{\beta}\,\mathrm{d}\Sigma_{\beta}. \tag{7.1}$$

Now starting from $c_{\alpha}(s,x) := \phi_{\alpha}(x)$, follow through the constructions of §6 with an arbitrarily chosen $A(s)$. This gives first $\omega(s,x)$ and $e_{\alpha}(s,x)$, and then $a(s)$ and $\nu(x)$ from (6.9) and $\psi(x)$ from (6.27) and (6.28). On substituting from (6.29) into (7.1), and using (5.11) and the result

$$\partial_{\alpha}\tau w^{\beta}\,\mathrm{d}\Sigma_{\beta} = \mathrm{d}\Sigma_{\alpha},$$

we now obtain
$$\langle J^{\alpha}, \phi_{\alpha}\rangle = \int \mathrm{d}s \int_{\Sigma(s)} (e_{\alpha}(s,x)\,\mathfrak{J}^{\alpha}w^{\beta} - \psi\mathfrak{J}^{\beta})\,\mathrm{d}\Sigma_{\beta} - \int aq\,\mathrm{d}s, \tag{7.2}$$

where
$$q(s) := \int_{\Sigma(s)} \mathfrak{J}^{\alpha}\,\mathrm{d}\Sigma_{\alpha}. \tag{7.3}$$

Note that although, in §6, we reserved the right to alter $c_{\alpha}(s,x)$ outside of the immediate neighbourhood of $S(s)$ when constructing ω and e_{α}, this does not alter the corresponding $\phi_{\alpha}(x)$ of (6.16) within S. Since by hypothesis $W \subset S$, this thus does not affect the validity of (7.2).

From (6.28), (6.27) and (6.11), the function $\psi(x)$ satisfies

$$\psi(z(s)) = 0, \quad \mathrm{d}\psi/\mathrm{d}u = u^{-1}v_{\lambda}\sigma^{\lambda\alpha}e_{\alpha}, \tag{7.4}$$

the second of these holding along all geodesic segments $x(u)$ lying in $S(s)$ having $x(0) = z(s)$ and u as an affine parameter. The integral

$$\int_{\Sigma(s)} (e_{\alpha}\mathfrak{J}^{\alpha}w^{\beta} - \psi\mathfrak{J}^{\beta})\,\mathrm{d}\Sigma_{\beta} \tag{7.5}$$

occurring in (7.2) thus depends on $e_{\alpha}(s,x)$ only through its value on $S(s)$. Now from (6.3) and (5.16) we see that the functional

$$e_{\alpha} \mapsto (2\pi)^{-4}\int \tilde{m}^{\lambda}(s,k)\,\tilde{E}_{\lambda}(z(s),k)\,\mathrm{D}k \tag{7.6}$$

is well defined if, in the terminology of § 5, $E_\lambda = Z^1(e_\alpha)$ in some neighbourhood of $\tilde{S}$. It is convenient also to require E_λ to satisfy (5.20), which we may do as it is compatible with (6.3). Moreover, the functional (7.6) also depends on $e_\alpha(s, x)$ only through its value on $S(s)$. We now adopt, as a provisional definition of the reduced moments of J^α, the hypothesis that (7.5) and (7.6) are actually identical. Hence

$$\int_{\Sigma(s)} (e_\alpha \mathfrak{J}^\alpha w^\beta - \psi \mathfrak{J}^\beta)\, \mathrm{d}\Sigma_\beta = (2\pi)^{-4} \int \tilde{m}^\lambda(s, k)\, \tilde{E}_\lambda(z(s), k)\, \mathrm{D}k \tag{7.7}$$

for arbitrary E_λ satisfying (5.20), and any $e_\alpha(s, x)$ satisfying (6.3) and given by $Z_1(E_\lambda)$ in some neighbourhood of $\delta(\tilde{S})$.

This provides our first link between the support of J^α and that of the functional m. It is not a complete link since E_λ in (7.7) is restricted by (5.20), but for such E_λ it shows that the right hand side of (7.7) depends on the value taken by E_λ in precisely that region of $T_{z(s)}(M)$ in which E_λ must be known in order to determine e_α and ψ on $\Sigma(s) \cap W$. Now to determine ψ at $x \in \Sigma(s) \cap W$, by (7.4) we need to know e_α along the whole geodesic segment joining $z(s)$ to x. However, by hypothesis this segment lies entirely in $\Sigma(s) \cap W$, so that if e_α is known throughout this cross-section, ψ is also completely determined on it. To so determine e_α, we need to know E_λ on

$$\mathrm{Exp}_{z(s)}^{-1}(\Sigma(s) \cap W).$$

The union of this over all s would thus form $\tilde{S}$, were it not for the restriction (5.20) on E_λ. We shall see below that this restriction can be removed, giving $\tilde{S}$ precisely in terms of J^α and the above geometric constructions.

By (5.21), any such E_λ can be written in the form $E_\lambda = X^\mu H_{\mu\lambda}$, where $H_{\mu\lambda} = H_{[\mu\lambda]}$, and conversely all E_λ of this form satisfy (5.20). We can use this to re-express (7.7) in a more convenient form. Choose $H_{\lambda\mu}$ arbitrarily, and let $h_{\alpha\beta} = h_{[\alpha\beta]}(s, x)$ equal $Z_1(H_{\lambda\mu})$ in some neighbourhood of $\delta(\tilde{S})$. Then by (5.15), $e_\alpha := \sigma^\beta h_{\beta\alpha}$ is given by $e_\alpha = Z_1(E_\lambda)$ in this neighbourhood. On using (5.24) and (5.27), (7.7) then gives

$$\int_{\Sigma(s)} (h_{\beta\gamma} \mathfrak{J}^\gamma \sigma^\beta w^\alpha - \psi \mathfrak{J}^\alpha)\, \mathrm{d}\Sigma_\alpha = (2\pi)^{-4} \int \tilde{Q}^{\lambda\mu} \tilde{H}_{\lambda\mu} \mathrm{D}k, \tag{7.8}$$

and (7.7) is equivalent to (7.8) holding for all such pairs $(h_{\alpha\beta}, H_{\lambda\mu})$. The results of § 5 now show that if any moments $Q^{\cdots}$ exist satisfying (7.8) and (5.28) as well as the symmetry and orthogonality conditions (5.32) and (5.36), then they are unique. This fixes all the moments of J^α except the monopole moment m^λ.

It was shown in § 5 of II that such $Q^{\cdots}$'s do exist, and explicit expressions were found for them as integrals of $\mathfrak{J}^\alpha$ over $\Sigma(s)$. Moreover, it was seen that for any choice of $m^\lambda(s)$, the support of the functional (7.6) is not enlarged when the restrictions (6.3) and (5.20) on e_α and E_λ are removed. For these moments the above discussion shows that we thus have

$$S(s) = \Sigma(s) \cap W, \quad S = W, \quad \text{and} \quad \tilde{S} = \textstyle\bigcup_s \mathrm{Exp}_{z(s)}^{-1}(\Sigma(s) \cap W). \tag{7.9}$$

We can now verify that, as a consequence of the restrictions placed above on W, l and n^λ, all the assumptions made about $S(s)$ and S are indeed satisfied. We can also verify (cf. (II, 6.3)) that, as imposed in § 5, $\tilde{m}^\lambda \to \infty$ as $k \to \infty$ no faster than a polynomial in k. In fact, it diverges linearly in k.

These proofs will not be repeated here; instead we study some other consequences of (7.7). From (5.1), (7.2) and (7.7) we get

$$\langle J^\alpha, \phi_\alpha \rangle = m[E_\lambda] - \int aq\, \mathrm{d}s. \tag{7.10}$$

Now in the present case, (6.1) gives

$$e_\alpha(s,x) = \phi_\alpha(x) - \partial_\alpha \omega(s,x) \tag{7.11}$$

for all x in some neighbourhood of $S(s)$. Let Ω be any scalar test function on TM over τ given by

$$\Omega = \omega \circ \delta, \tag{7.12}$$

as in (3.2), in some neighbourhood of $\delta(\tilde{S})$. Then from (3.10) and (3.11),

$$\nabla_\alpha \omega = -\sigma^{\lambda}{}_{.\alpha} \nabla_{*\lambda} \Omega \tag{7.13}$$

in this neighbourhood. Hence if we choose

$$Z_{1}{}^{\lambda}{}_{.\alpha} = -\sigma^{\lambda}{}_{.\alpha}, \tag{7.14}$$

which satisfies (5.15) as required, we have $\partial_\alpha \omega = Z_1(\nabla_{*\lambda}\Omega)$ in some neighbourhood of $\delta(\tilde{S})$, and so from (7.11), $\phi_\alpha = Z_1(\Phi_\lambda)$ also, where

$$\Phi_\lambda(z,X) := E_\lambda + \nabla_{*\lambda}\Omega. \tag{7.15}$$

This choice of Z_1 is equivalent to using (5.12) instead of (5.13) for e_α.

Substitution from (7.15) into (7.10), with the use of (5.4) and (7.13), now yields

$$\langle J^\alpha, \phi_\alpha \rangle = m[\Phi_\lambda] - \int (aq + m^\lambda \partial_\lambda \omega)\, ds. \tag{7.16}$$

But from (7.11) and (6.12), $\partial_\lambda \omega = \phi_\lambda(z(s))$. On using this and (6.18), we can thus rewrite (7.16) as

$$\langle J^\alpha, \phi_\alpha \rangle = m[\Phi_\lambda] + \int \phi_\lambda (qv^\lambda - m^\lambda)\, ds + \int \frac{dq}{ds} A\, ds, \tag{7.17}$$

where we have used the fact that $A(s)$ has compact support to integrate the last term by parts.

Note that in (7.17), all mention of ω and e_α has disappeared. It is true that Φ_λ has been constructed from ω and e_α, but it is completely determined in the neighbourhood of $\tilde{S}$ by its relationship to ϕ_α, and since $\tilde{S}$ is the support of m, this shows that $m[\Phi_\lambda]$ is completely determined by ϕ_α. Now the arbitrary function $A(s)$ occurs in (7.17) only in the last term. This term must thus vanish identically, giving

$$dq/ds = 0. \tag{7.18}$$

If, in addition, we take

$$m^\lambda(s) := q(s)\, v^\lambda, \tag{7.19}$$

thus defining the only remaining moment of J^α, (7.16) gives

$$\langle J^\alpha, \phi_\alpha \rangle = m[\Phi_\lambda]. \tag{7.20}$$

In (7.18) we have recovered, by a tortuous route, the law of conservation of total charge. The other result, (7.20), forms the basis of our final definition of the reduced moments of J^α. We now set this out as a theorem: Let J^α be a C^1 vector field on M whose support satisfies the conditions given at the beginning of this section, and let a world line l and timelike unit vector field $n^\lambda(s)$ along l be chosen also in accordance with these conditions. Let $\tilde{m}^\lambda(s,k)$ be defined by (4.15), in terms of certain tensor fields $m^{\cdots}$ along l satisfying (4.1), (4.4) and (4.5), and suppose that for each fixed s, $\tilde{m}^\lambda \to \infty$ as $k \to \infty$ no faster than a polynomial in k. From it, construct the functional $m[E_\lambda]$ according to (5.1). Now suppose that there exists a bounded closed set

$$S' \subset N \subset TM$$

such that if ϕ_α is any vectorial test function (i.e. a C^∞ function of compact support) on M, then (7.20) is satisfied for all test fields Φ_λ on TM over τ which satisfy

$$\Phi_\lambda = \mathrm{Exp}^A \phi_\alpha \tag{7.21}$$

in some neighbourhood of S'. Suppose further that no closed subset of S' also has this property. Then

(i) We must have
$$\nabla_\alpha J^\alpha = 0. \tag{7.22}$$

Conversely, if J^α satisfies this, then moments $m^{\cdots}$ and a region $S' \subset N$ do exist having all the required properties.

(ii) $S' = S''$, where
$$S'' := \bigcup_s \mathrm{Exp}^{-1}_{z(s)}(\Sigma(s) \cap W). \tag{7.23}$$

(iii) The $m^{\cdots}$'s are uniquely determined.

(iv) $m^\lambda(s)$ is given by (7.19) and (7.3), and satisfies the charge conservation equation (7.18).

(v) The higher moments are given by equations (5.66), (5.62), (5.60) and (5.58) of II as explicit integrals over $\Sigma(s)$.

(vi) There are no algebraic or differential relations between the moments as a consequence of (7.22), other than (7.18).

Before this is proved, a few comments on it will be made.

(*a*) Result (i) is essentially a consequence of the process of reduction discussed in the introduction. Although moments can be constructed for an arbitrary vector field by using the explicit integral expressions of (iv) and (v) above, they will in general neither satisfy (7.20), nor contain sufficient information to reconstruct the original vector field. In such a case it is necessary to use the complete moments, defined in II, in order to have a set from which J^α can be reconstructed. Result (vi) shows that the reduction process is complete.

(*b*) Whereas the provisional definition (7.7) apparently builds in the hypersurfaces $\Sigma(s)$ as the surfaces of integration over which the moments will be defined, this is no longer the case in (7.20). Result (ii) shows that this choice of hypersurfaces is not a free one; it is a consequence of the orthogonality conditions imposed on the moments. Since a change of n^λ will thus change the surfaces of integration, it is clear that there can be no algebraic relation between the reduced moments for different choices of n^λ, unless J^α is assumed to be analytic.

(*c*) The requirement that no subset S' has a similar property is not a trivial one; it was shown in §3 of II that unless this is imposed, it is possible in certain circumstances to lose the uniqueness of the corresponding moments. The restriction to closed subsets of TM is to prevent the trivial omission of isolated points yielding subsets of S' which nevertheless have identical neighbourhoods to S', and which thus impose identical restrictions on the relationship between ϕ_α and Φ_λ.

We turn now to the proof of this theorem. We first prove together both (ii) and the converse part of (i). As before, let $\tilde{S}$ denote the support of m. Then (7.20) can only be valid under the given conditions if $\tilde{S} \subset S'$, for otherwise the value of the right hand side could be altered without changing ϕ_α. But in the notation of §5, also $\tilde{S} \subset \tilde{\Sigma}$, and hence $\tilde{S} \subset S' \cap \tilde{\Sigma}$. Similarly, $W \subset \mathrm{Exp}\,\tilde{S}$, for otherwise the value of the left hand side of (7.20) could be altered without changing that of the right hand side. Hence $W \subset \mathrm{Exp}\,(S' \cap \tilde{\Sigma})$, and so for each s,

$$W \cap \Sigma(s) \subset \mathrm{Exp}_{z(s)}(S' \cap \tilde{\Sigma}(s)),$$

which by (7.23) implies $S'' \subset S'$. But we have already seen that moments exist satisfying (7.20) under the required conditions when S' is the $\tilde{S}$ of (7.9), which is precisely S''. The minimality condition on S' thus ensures that $S' = S''$ is the only solution. Hence necessarily $\tilde{S} = S''$ also, and $S(s)$ and S are as given by (7.9). As we saw above, we are now assured that the restrictions placed on $\tilde{S}$ in § 5 are satisfied, and so all the results of §§ 5 and 6 are valid.

Result (iii), the uniqueness of the moments, will be proved by deducing both (7.8) and (7.19) from (7.20). Let $h_{\alpha\beta}(s,x)$ and $H_{\lambda\mu}(z,X)$ satisfy the conditions imposed on them in (7.8), with Z_1 given by (7.14). Put

$$e_\alpha = \sigma^\beta h_{\beta\alpha} \quad \text{and} \quad E_\mu = X^\lambda H_{\lambda\mu}, \tag{7.24}$$

and construct from e_α the functions $c_\alpha(s,x)$ and $\omega(s,x)$ of (6.1), and $\phi_\alpha(x)$ of (6.16), according to the methods of § 6. Pick a representative $\Omega(z,X)$ of ω as in (7.12), and Φ_λ of ϕ_α as in (7.21), and put

$$C_\lambda(z,X) := E_\lambda + \nabla_{*\lambda}\Omega. \tag{7.25}$$

Then $c_\alpha = \operatorname{Exp}_A C_\lambda$ in a neighbourhood of $\delta(\tilde{S})$. Hence with (7.21), (6.5) and (5.39) we have

$$m[\Phi_\lambda] = m[C_\lambda], \tag{7.26}$$

and from (7.25) and (5.4),
$$m[C_\lambda] = m[E_\lambda] + \int \mathrm{d}s\, m^\lambda \partial_\lambda \omega. \tag{7.27}$$

Use (6.9) on the last term in this, and substitute from it and (7.26) into the right hand side of (7.20). On transforming its left hand side using (7.1) and (6.29), we obtain

$$m[E_\lambda] + \int \mathrm{d}s\, m^\lambda(\partial_\lambda \nu - a\partial_\lambda \tau) = \int \mathrm{d}s \int_{\Sigma(s)} [\mathfrak{J}^\alpha e_\alpha w^\beta - \psi \mathfrak{J}^\beta]\, \mathrm{d}\Sigma_\beta - \int \nu \nabla_\alpha \mathfrak{J}^\alpha \mathrm{d}^4x - \int aq\, \mathrm{d}s. \tag{7.28}$$

Now in this, $a(s)$ and $\nu(x)$ are arbitrary. By first putting $a = 0$, $\nu = 0$ we obtain

$$\int \mathrm{d}s \left\{ \int_{\Sigma(s)} [\mathfrak{J}^\alpha e_\alpha w^\beta - \psi \mathfrak{J}^\beta]\, \mathrm{d}\Sigma_\beta - (2\pi)^{-4} \int \tilde{m}^\lambda \tilde{E}_\lambda \,\mathrm{D}k \right\} = 0, \tag{7.29}$$

and then by putting $a = 0$ and $\nu = 0$ separately in turn, we get also

$$\int \mathrm{d}s\, m^\lambda \partial_\lambda \nu = -\int \nu \nabla_\alpha \mathfrak{J}^\alpha \mathrm{d}^4x \tag{7.30}$$

and
$$q(s) = m^\lambda(s)\, \partial_\lambda \tau. \tag{7.31}$$

Since $\nabla_\alpha \mathfrak{J}^\alpha$ is continuous by hypothesis, (7.30) can only hold for all $\nu(x)$ if both sides vanish identically. The vanishing of the right hand side then completes the proof of (i). The vanishing of the left hand side is equation (II, 5.40), which was shown in II to imply that m^λ has the form $m^\lambda = kv^\lambda$, for some constant k. On putting this into (7.31), we get $k = q(s)$, which is result (iv). Now the relationships between $h_{\alpha\beta}$, ψ and $H_{\lambda\mu}$ are all maintained if we replace them by

$$\theta(s)\, h_{\alpha\beta}(s,x), \quad \theta(\tau(x))\, \psi(x) \quad \text{and} \quad \theta(\tau(z))\, H_{\lambda\mu}(z,X)$$

respectively, where $\theta(s)$ is an arbitrary C^∞ function of s. In the s-integrand of (7.29), all the θ-factors become $\theta(s)$, and by using the arbitrariness of θ it then follows that the curly bracket in (7.29) must vanish identically. This gives (7.7), from which (7.8), and consequently results (iii) and (v), follow as discussed above. Finally to prove (vi), we put $\phi_\alpha(x) = \partial_\alpha \omega(x)$ in (7.20). The corresponding Φ_λ can then be taken in the form $\nabla_{*\lambda}\Omega$, and so (7.20) and (5.4) give (cf. (7.16))

$$\langle \partial_\alpha J^\alpha, \omega \rangle = -\int m^\lambda \partial_\lambda \omega\, \mathrm{d}s,$$

which vanishes by (7.18) and (7.19). Hence (7.18) is the only restriction on the moments needed to ensure (7.22), and hence it is the only restriction implied by (7.22), as required.

We note for later use that if ψ is defined using (6.8) rather than (6.11), so that (7.4) is replaced by

$$\psi(z(s)) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}u}\psi = -\dot{x}^{\alpha}\frac{\partial}{\partial s}e_{\alpha} \tag{7.32}$$

in $S(s)$ along the appropriate geodesic segments, then (7.29) is valid for arbitrary e_{α}, and not just for those satisfying $\sigma^{\alpha}e_{\alpha} = 0$. Its derivation immediately following (7.24) is still valid in this case as the relevant results from §6 did not use this restriction except in the deduction of (6.11) from (6.8). However, we can no longer deduce (7.7) from (7.29), since replacement of e_{α} by $\theta(s)\,e_{\alpha}(s,x)$ does not now correspond to $\psi \to \theta(\tau(x))\,\psi(x)$. This result was also proved in §5 of II.

8. Definition of the reduced moments of $T^{\alpha\beta}$

We now follow through this same procedure for $T^{\alpha\beta}$, to obtain first a provisional definition of the reduced moments, and then a more satisfactory final definition. Let $\phi_{\alpha\beta} = \phi_{(\alpha\beta)}(x)$ be a tensorial test function on M. Starting from

$$c_{\alpha\beta}(s,x) := \phi_{\alpha\beta}(x), \tag{8.1}$$

follow through the constructions of §6 with the tensors $A_{\kappa}(s)$ and $B_{\kappa\lambda} = B_{[\kappa\lambda]}(s)$ of (6.55) being chosen arbitrarily. This first gives us $e_{\alpha\beta}(s,x)$ and $\omega_{\alpha}(s,x)$, from which we construct $\nu_{\alpha}(x)$ by (6.39), $a_{\kappa}(s)$ and $b_{\kappa\lambda}(s)$ by (6.41), and $\psi_{\alpha}(x)$ by (6.42) and (6.45). Then by substituting from (6.44) and (6.42) into (5.9) and using (5.11), we obtain

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta}\rangle = \int \mathrm{d}s \int_{\Sigma(s)} (e_{\alpha\beta}\mathfrak{T}^{\alpha\beta}w^{\gamma} - \psi_{\alpha}\mathfrak{T}^{\alpha\gamma})\,\mathrm{d}\Sigma_{\gamma} - \int (a_{\kappa}\bar{p}^{\kappa} + \tfrac{1}{2}b_{\kappa\lambda}\bar{S}^{\kappa\lambda})\,\mathrm{d}s + \int F^{\alpha\beta}\mathfrak{J}_{\beta}\nu_{\alpha}\,\mathrm{d}^4x, \tag{8.2}$$

where

$$\left.\begin{aligned} \bar{p}^{\kappa}(s) &:= \int_{\Sigma(s)} K_{\alpha}^{\cdot\kappa}\mathfrak{T}^{\alpha\beta}\,\mathrm{d}\Sigma_{\beta} \\ \bar{S}^{\kappa\lambda}(s) &:= 2\int_{\Sigma(s)} H_{\alpha}^{\cdot[\kappa}\sigma^{\lambda]}\mathfrak{T}^{\alpha\beta}\,\mathrm{d}\Sigma_{\beta}. \end{aligned}\right\} \tag{8.3}$$

and

When $F_{\alpha\beta} = 0$, the definitions (8.3) agree with those of the momentum and spin tensors p^{κ} and $S^{\kappa\lambda}$ given by equations (I, 5.1) and (I, 5.2), but if $F_{\alpha\beta} \neq 0$, p^{κ} and $S^{\kappa\lambda}$ contain electromagnetic contributions which are absent in (8.3). However, when $F_{\alpha\beta} \neq 0$, (8.2) differs from the analogous equation (7.2) for $\mathfrak{J}^{\alpha}$ by still containing $\nu_{\alpha}(x)$. These two discrepancies are connected, as will now be shown.

On using (6.39), the final term in (8.2) can be written as

$$\int F^{\alpha\beta}\mathfrak{J}_{\beta}\nu_{\alpha}\,\mathrm{d}^4x = \int \mathrm{d}s \int_{\Sigma(s)} F^{\alpha\beta}\omega_{\alpha}(s,x)\,\mathfrak{J}_{\beta}w^{\gamma}\,\mathrm{d}\Sigma_{\gamma}. \tag{8.4}$$

Now define

$$e_{\beta}(s,x) := F_{\alpha\beta}\omega^{\alpha}. \tag{8.5}$$

From it construct $\psi(x)$ as in (7.32), and pick an $E_{\lambda}(z,X)$ given by $\mathrm{Exp}^{A}\,e_{\alpha}$ in some neighbourhood of $\delta(\tilde{S})$. Then on using (7.29) and (5.1), we can put (8.4) in the form

$$\int F^{\alpha\beta}\mathfrak{J}_{\beta}\nu_{\alpha}\,\mathrm{d}^4x = m[E_{\lambda}] + \int \mathrm{d}s \int \psi\mathfrak{J}^{\alpha}\,\mathrm{d}\Sigma_{\alpha}. \tag{8.6}$$

We next separate out the dependence of ψ on a_{κ} and $b_{\kappa\lambda}$, using (7.32), (8.5), (6.42) and (6.45). Choose bitensors $\Psi_{\kappa}(s,x)$ and $\Phi_{\kappa}(s,x)$ of compact support on each tangent space $T_{z(s)}(M)$ and satisfying

$$\Psi_{\kappa}(s,z(s)) = 0, \quad \mathrm{d}\Psi_{\kappa}(s,x)/\mathrm{d}u = \dot{x}^{\beta}K^{\alpha}_{\cdot\kappa}F_{\alpha\beta}, \tag{8.7}$$

$$\Phi_{\kappa}(s,z(s)) = 0, \quad \mathrm{d}(u\Phi_{\kappa})/\mathrm{d}u = u\dot{x}^{\beta}H^{\alpha}_{\cdot\kappa}F_{\alpha\beta}, \tag{8.8}$$

along all geodesic segments $x^\alpha(u)$ having $x(0) = z(s)$, u as an affine parameter, and lying in some neighbourhood of $S(s)$. These definitions agree with those of equations (I, 3.14) and (I, 3.15). Further, choose a $\chi(x)$ of compact support and satisfying

$$\chi(z(s)) = 0 \quad \text{and} \quad \mathrm{d}\chi/\mathrm{d}u = \dot{x}^\alpha F_{\alpha\beta}\psi^\beta \tag{8.9}$$

along those geodesic segments lying in $S(s)$. Then ψ is given on $S(s)$ by

$$\psi(x) = \chi(x) - a_\kappa(s)\,\Psi^\kappa(s,x) - b_{\kappa\lambda}(s)\,\sigma^\lambda(z(s),x)\,\Phi^\kappa(s,x). \tag{8.10}$$

If we now substitute this into (8.6), and the result into (8.2), we obtain

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta}\rangle = \int \mathrm{d}s \int_{\Sigma(s)} (e_{\alpha\beta}\mathfrak{T}^{\alpha\beta}w^\gamma - \psi_\alpha\mathfrak{T}^{\alpha\gamma} + \chi\mathfrak{J}^\gamma)\,\mathrm{d}\Sigma_\gamma + m[E_\lambda] - \int (a_\kappa p^\kappa + \tfrac{1}{2}b_{\kappa\lambda}S^{\kappa\lambda})\,\mathrm{d}s, \tag{8.11}$$

where
$$p^\kappa(s) := \int_{\Sigma(s)} (K_{\alpha}^{:\kappa}\mathfrak{T}^{\alpha\beta} + \Psi^\kappa\mathfrak{J}^\beta)\,\mathrm{d}\Sigma_\beta \tag{8.12}$$

and
$$S^{\kappa\lambda}(s) := 2\int_{\Sigma(s)} \sigma^{[\lambda}(H_{\alpha}^{:\kappa]}\mathfrak{T}^{\alpha\beta} + \Phi^{\kappa]}\mathfrak{J}^\beta)\,\mathrm{d}\Sigma_\beta. \tag{8.13}$$

These p^κ and $S^{\kappa\lambda}$ are precisely the momentum and spin tensors defined in §5 of I.

Equation (8.11) is the true analogue of (7.2), and we use it to suggest the appropriate provisional definition of the moments of $T^{\alpha\beta}$. By analogy with (7.7), we require

$$\int_{\Sigma(s)} (e_{\alpha\beta}\mathfrak{T}^{\alpha\beta}w^\gamma - \psi_\alpha\mathfrak{T}^{\alpha\gamma} + \chi\mathfrak{J}^\gamma)\,\mathrm{d}\Sigma_\gamma = (2\pi)^{-4}\int \tilde{I}^{\lambda\mu}\tilde{E}_{\lambda\mu}\,\mathrm{D}k \tag{8.14}$$

for arbitrary symmetric $E_{\lambda\mu}(z,X)$ satisfying (5.20), and any $e_{\alpha\beta}(s,x)$ satisfying (5.19) and given by $Z_2(E_{\lambda\mu})$ in some neighbourhood of $S(s)$. As before, we can use (5.22) to put (8.14) in an equivalent form. Let $H_{\kappa\lambda\mu\nu}(z,X)$ be an arbitrary test field on TM over τ satisfying the symmetry conditions of (5.22), and let $h_{\alpha\beta\gamma\delta}(s,x) = Z_2(H_{\kappa\lambda\mu\nu})$ in some neighbourhood of $\delta(\tilde{S})$. Put

$$E_{\kappa\lambda} := X^\mu X^\nu H_{\kappa\mu\lambda\nu}, \quad e_{\alpha\beta} := \sigma^\gamma\sigma^\delta h_{\alpha\gamma\beta\delta}. \tag{8.15}$$

Then on using (5.26) and (5.27), we get (8.14) in the form

$$\int_{\Sigma(s)} (h_{\alpha\gamma\beta\delta}\sigma^\gamma\sigma^\delta\,\mathfrak{T}^{\alpha\beta}w^\epsilon - \psi_\alpha\mathfrak{T}^{\alpha\epsilon} + \chi\mathfrak{J}^\epsilon)\,\mathrm{d}\Sigma_\epsilon = (2\pi)^{-4}\int \tilde{J}^{\kappa\lambda\mu\nu}\tilde{H}_{\kappa\lambda\mu\nu}\,\mathrm{D}k. \tag{8.16}$$

As was the case for (7.8), if any $J^{\cdots}$'s exist satisfying (8.16) with (5.29) then they are unique and give all the moments of $T^{\alpha\beta}$ except for the monopole and dipole moments $I^{\lambda\mu}$ and $I^{\kappa\lambda\mu}$. We shall show in §9 that such $J^{\cdots}$'s do exist, and that for any choice of $I^{\lambda\mu}$ and $I^{\kappa\lambda\mu}$, the domain of dependence of the right hand side of (8.14) on $E_{\lambda\mu}$ is not enlarged when the restriction (5.20) is removed. The support $\tilde{S}$ of $I[E_{\lambda\mu}]$ is thus given by (7.9), just as was the case for $m[E_\lambda]$. For the present we shall assume these results and shall continue to follow the method leading from (7.8) to (7.20).

On substituting from (8.14) into (8.11) and using (5.2), we obtain

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta}\rangle = I[E_{\lambda\mu}] + m[E_\lambda] - \int (a_\kappa p^\kappa + \tfrac{1}{2}b_{\kappa\lambda}S^{\kappa\lambda})\,\mathrm{d}s. \tag{8.17}$$

Now from (8.1) and (6.2) we have

$$e_{\alpha\beta}(s,x) = \phi_{\alpha\beta}(x) - \nabla_{(\alpha}\omega_{\beta)}(s,x) \tag{8.18}$$

in some neighbourhood of $S(s)$. We wish to find an equivalent relation between $E_{\lambda\mu}$ and the corresponding images under Z^2 of $\phi_{\alpha\beta}$ and ω_α. Following the analogy with §7, it should be at this point that it becomes clear what choice to make for $Z_{2\cdot\alpha}^{\lambda}$. However, in this case both choices (5.12) and (5.13) will still seem equally natural, while it will no longer be possible to work with a general Z_2. We shall thus continue to work separately with both these choices for the time being, labelling corresponding equations by a suffix (a) or (b). For later use, we shall not deal with (8.18) directly, but instead with the original (6.2), of which (8.18) is a special case.

For a fixed value of s, we can express (6.2) as

$$c_{\alpha\beta} = e_{\alpha\beta} + \tfrac{1}{2}L_\omega g_{\alpha\beta} \tag{8.19a}$$

or as

$$c^{\alpha\beta} = e^{\alpha\beta} - \tfrac{1}{2}L_\omega g^{\alpha\beta}. \tag{8.19b}$$

If we decompose ω^α according to (6.69), we see from (6.70), (3.26) and (3.22) that L_ξ commutes with Exp_A. Also, since the λ^α of (6.69) is the solution of (6.54) with the initial conditions (6.55) having $A_\kappa = 0$ and $B_{\kappa\lambda} = 0$, we see that this λ^α satisfies (3.20) and hence L_λ satisfies (3.18). Now choose tensorial test functions on TM over τ satisfying

$$E_{\lambda\mu} = \mathrm{Exp}^A e_{\alpha\beta} \quad \text{and} \quad G_{\lambda\mu} = \mathrm{Exp}^A g_{\alpha\beta}, \tag{8.20a}$$

or

$$E^{\lambda\mu} = \mathrm{Exp}^A e^{\alpha\beta} \quad \text{and} \quad G^{\lambda\mu} = \mathrm{Exp}^A g^{\alpha\beta}, \tag{8.20b}$$

and also

$$\Lambda^\kappa = \mathrm{Exp}^A \lambda^\alpha, \tag{8.21}$$

in some neighbourhood of $\tilde{S}$, and such that $E_{\lambda\mu}$ satisfies (5.20). Then if we put

$$C_{\lambda\mu} = E_{\lambda\mu} + \tfrac{1}{2}L_\xi G_{\lambda\mu} + \tfrac{1}{2}L_\Lambda G_{\lambda\mu}, \tag{8.22a}$$

or

$$C^{\lambda\mu} = E^{\lambda\mu} - \tfrac{1}{2}L_\xi G^{\lambda\mu} - \tfrac{1}{2}L_\Lambda G^{\lambda\mu}, \tag{8.22b}$$

we see that $C_{\lambda\mu} = \mathrm{Exp}^A c_{\alpha\beta}$ or $C^{\lambda\mu} = \mathrm{Exp}^A c^{\alpha\beta}$ respectively, in this neighbourhood. Now

$$L_\Lambda G_{\lambda\mu} = \Lambda^\kappa \nabla_{*\kappa} G_{\lambda\mu} + 2G_{\kappa(\lambda}\nabla_{*\mu)}\Lambda^\kappa, \tag{8.23}$$

and if we put

$$M_\lambda := G_{\lambda\mu}\Lambda^\mu, \tag{8.24}$$

this becomes

$$L_\Lambda G_{\lambda\mu} = 2\nabla_{*(\lambda}M_{\mu)} - \Lambda^\kappa \nabla_{*\{\lambda}G_{\kappa\mu\}}. \tag{8.25}$$

The terms involving derivatives of Λ^κ thus occur in (8.22a) in the form of a symmetrized derivative $\nabla_{*(\lambda}M_{\mu)}$, to which (5.5) may be applied. On lowering the indices in (8.22b), however, we find that derivatives of Λ_μ occur on the right hand side in the combination

$$g_{\lambda\rho}g_{\mu\sigma}L_\Lambda G^{\rho\sigma} = \Lambda^\kappa \nabla_{*\kappa} G_{\lambda\mu} - 2(\nabla_{*\kappa}\Lambda_{(\lambda})\, G_{\mu)}^{\cdot\,\kappa}, \tag{8.26}$$

which cannot be similarly converted into a symmetrized derivative. On these grounds we adopt the choice (a), and shall from now on work solely with this.

To use (5.5), we see that we need to know $\nabla_{*\lambda}M_\mu$ and $\nabla_{*\kappa\lambda}M_\mu$ when $X^\mu = 0$. To find these, we first deduce from (8.20a) and (3.16) that

$$\hat{G}_{\lambda\mu}(z,x) = H_{\alpha\lambda}H^{\alpha}_{\cdot\mu}. \tag{8.27}$$

We then use (3.11) and the coincidence limits of derivatives of $H_{\alpha\lambda}$ derived in the appendix of II. The result simplifies since $\lambda^\kappa(z,z) = 0$, and we finally obtain

$$\nabla_{*\kappa}M_\mu = \nabla_\kappa \lambda_\mu, \quad \nabla_{*\kappa\lambda}M_\mu = \nabla_{\kappa\lambda}\lambda_\mu \quad \text{when} \quad X^\lambda = 0. \tag{8.28}$$

The convention on the right hand sides is again that $\nabla_\kappa \lambda_\mu$ denotes the coincidence limit of $\nabla_\alpha \lambda_\beta(x,z)$ as $x \to z$. These can be further evaluated using (6.54), (6.55) and the definition of λ^α in (6.69), to give

$$\nabla_{*\kappa}M_\mu = c_{\kappa\mu}, \quad \nabla_{*\kappa\lambda}M_\mu = \nabla_{\{\kappa}c_{\mu\lambda\}} \quad \text{when} \quad X^\lambda = 0. \tag{8.29}$$

Now return to the choice (8.1) for $c_{\alpha\beta}$, and correspondingly write $C_{\lambda\mu}$ as $\Phi_{\lambda\mu}$. Substitute from (8.22*a*) and (8.25) into (8.17) and use (5.5) with (8.29) on the term $I[\nabla_{*(\lambda}M_{\mu)}]$. Also, since it is A_κ and $B_{\kappa\lambda}$ rather than a_κ and $b_{\kappa\lambda}$ that are arbitrary in our construction, substitute from (6.59) and (6.62) into the final term of (8.17). If we then integrate by parts the terms in $\delta A_\kappa/\mathrm{d}s$ and $\delta B_{\kappa\lambda}/\mathrm{d}s$, we finally obtain

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta}\rangle = I[\Phi_{\lambda\mu}+\tfrac{1}{2}A^\kappa\nabla_{*\{\lambda}G_{\kappa\mu\}}-\tfrac{1}{2}L_\xi G_{\lambda\mu}]+m[E_\lambda]+\int \mathrm{d}s\{A_\kappa(\delta p^\kappa/\mathrm{d}s+\tfrac{1}{2}R_{\lambda\mu\nu}{}^{\kappa}S^{\lambda\mu}v^\nu)$$
$$+\tfrac{1}{2}B_{\kappa\lambda}(\delta S^{\kappa\lambda}/\mathrm{d}s-2p^{[\kappa}v^{\lambda]})\}-\int \mathrm{d}s\{(I^{\lambda\mu}-p^{(\lambda}v^{\mu)})\,\phi_{\lambda\mu}+(I^{\kappa\lambda\mu}-S^{\kappa(\lambda}v^{\mu)})\,\nabla_\kappa\phi_{\lambda\mu}\}. \tag{8.30}$$

By analogy with (7.19), we now choose

$$I^{\lambda\mu} = p^{(\lambda}v^{\mu)} \quad\text{and}\quad I^{\kappa\lambda\mu} = S^{\kappa(\lambda}v^{\mu)}, \tag{8.31}$$

which satisfy identically the symmetry conditions

$$I^{\lambda\mu} = I^{(\lambda\mu)}, \quad I^{\kappa\lambda\mu} = I^{\kappa(\lambda\mu)}, \quad I^{(\kappa\lambda\mu)} = 0 \tag{8.32}$$

required by (4.7) and (4.10). This completes the definition of the moments, and makes the final integral in (8.30) vanish.

We next want to use the arbitrariness of A_κ and $B_{\kappa\lambda}$ to give the equations of motion. These arbitrary functions occur in (8.30) in a much more involved way than the occurrence of $A(s)$ in (7.17), as they occur in the arguments of I and m as well as in the explicit integral over s. They have been separated off in the argument of I into the term $-\tfrac{1}{2}L_\xi G_{\lambda\mu}$. We must next perform a similar separation in the argument E_λ of m.

From (8.5) and (6.69),

$$e_\beta = \lambda^\alpha F_{\alpha\beta}+\xi^\alpha F_{\alpha\beta}, \tag{8.33}$$

and $E_\lambda = \mathrm{Exp}^A e_\alpha$. As in our treatment of (8.19), we shall treat each term in (8.33) in a different way under the action of Exp^A. We first introduce, for each z, a particular choice of vector potential for $F_{\alpha\beta}$. In doing so, we use for the first time the Maxwell field equation

$$\nabla_{[\gamma}F_{\alpha\beta]} = 0. \tag{8.34}$$

Let $A_\alpha(z,x)$ be the vector potential satisfying

$$\sigma^\alpha(z,x)\,A_\alpha = 0 \tag{8.35}$$

in addition to

$$F_{\alpha\beta} = \nabla_\beta A_\alpha - \nabla_\alpha A_\beta. \tag{8.36}$$

By multiplying (8.36) by σ^α and using the techniques of the proof of (6.37), we then find that

$$\mathrm{d}(uH^{\alpha}{}_{\cdot\lambda}A_\alpha)/\mathrm{d}u = u\dot{x}^\beta H^{\alpha}{}_{\cdot\lambda}F_{\alpha\beta}. \tag{8.37}$$

There is a unique solution of this which is finite at $x = z$, and on comparison with (8.8) it is seen to be given by

$$H^{\alpha}{}_{\cdot\lambda}A_\alpha(z(s),x) = \Phi_\lambda(s,x). \tag{8.38}$$

Hence, on using the convention of § 3 of not distinguishing notationally between the fields Φ and $\hat{\Phi}$ of (3.2), we have

$$\Phi_\lambda = \mathrm{Exp}^A A_\alpha, \tag{8.39}$$

which with (8.36) and (3.11) gives $\mathrm{Exp}^A F_{\alpha\beta} = -2\nabla_{*[\kappa}\Phi_{\lambda]}$. (8.40)

From (8.21) and (8.40), we thus see that the contribution to E_λ from the λ^α-term in (8.33) is $-2A^\kappa\nabla_{*[\kappa}\Phi_{\lambda]}$.

We turn next to the ξ^α-term. This may be rewritten as

$$\xi^\alpha F_{\alpha\beta} = \nabla_\beta(\xi^\alpha A_\alpha) + \xi^\kappa \nabla_\kappa A_\beta - L_\xi A_\beta, \tag{8.41}$$

on remembering that in $L_\xi A_\beta$ we must treat A_β as a vector at x and as a scalar at z. From (8.39), and (3.22) with $\lambda^\alpha = \xi^\alpha$, we have

$$\mathrm{Exp}^A L_\xi A_\beta = L_\xi \Phi_\lambda. \tag{8.42}$$

It thus remains only to deal with the first two terms on the right hand side of (8.41). We do this by showing that there exists a scalar field $Z(z, X)$ on TM such that

$$\mathrm{Exp}^A [\nabla_\beta(\xi^\alpha A_\alpha) + \xi^\kappa \nabla_\kappa A_\beta] = \nabla_{*\lambda} Z. \tag{8.43}$$

The corresponding contribution to $m[E_\lambda]$ can then be evaluated using (5.4). To prove (8.43), we note that any other vector potential can be obtained from $A_\alpha(z, x)$ by adding the gradient of a scalar. There must thus exist a biscalar $\theta(z, x)$ such that $(A_\alpha(z, x) + \partial_\alpha \theta(z, x))$ is independent of z. Hence $\nabla_\kappa A_\alpha = -\nabla_{\kappa\alpha}\theta$. Since $\xi^\kappa(z, z)$ is independent of x, we thus have

$$\nabla_\beta(\xi^\alpha A_\alpha) + \xi^\kappa \nabla_\kappa A_\beta = \nabla_\beta(\xi^\alpha A_\alpha - \xi^\kappa \nabla_\kappa \theta), \tag{8.44}$$

and so (8.43) is satisfied with

$$Z = \mathrm{Exp}^A (\xi^\alpha A_\alpha - \xi^\kappa \nabla_\kappa \theta), \tag{8.45}$$

as required.

In using (8.43) in (5.4), we need to know the coincidence limit of (8.44) as $x \to z$. Coincidence limits will be denoted by diamond brackets. Then $\langle A_\alpha \rangle = 0$ by (8.35), and so on using (II, A 15) we see that also

$$\langle L_\xi A_\beta \rangle = \xi^\kappa \nabla_\kappa \langle A_\beta \rangle + \langle A_\alpha \nabla_\beta \xi^\alpha \rangle = 0. \tag{8.46}$$

Hence from (8.41),

$$\langle \nabla_\beta(\xi^\alpha A_\alpha) + \xi^\kappa \nabla_\kappa A_\beta \rangle = \langle \xi^\alpha F_{\alpha\beta} \rangle \tag{8.47}$$

as required. On piecing together the above results, we finally obtain

$$m[E_\lambda] = m[2A^\kappa \nabla_{*[\lambda} \Phi_{\kappa]} - L_\xi \Phi_\lambda] + \int \mathrm{d}s\, q v^\lambda \xi^\kappa F_{\kappa\lambda}, \tag{8.48}$$

in which the desired separation has been achieved.

Use $\xi^\kappa = A^\kappa$ in (8.48) and substitute it into (8.30). Since A^κ and $B^{\kappa\lambda}$ are arbitrary, the resulting expression separates into two equations analogous to (7.18) and (7.20). We obtain

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta} \rangle = I[\Phi_{\lambda\mu} + \tfrac{1}{2} A^\kappa \nabla_{*\{\lambda} G_{\kappa\mu\}}] + m[2A^\kappa \nabla_{*[\lambda} \Phi_{\kappa]}] \tag{8.49}$$

and

$$A_\kappa\left(\frac{\delta}{\mathrm{d}s} p^\kappa + \tfrac{1}{2} R_{\lambda\mu\nu}{}^{\kappa} S^{\lambda\mu} v^\nu + q F^\kappa{}_\lambda v^\lambda\right) + \tfrac{1}{2} B_{\kappa\lambda}\left(\frac{\delta}{\mathrm{d}s} S^{\kappa\lambda} - 2p^{[\kappa} v^{\lambda]}\right)$$
$$= (2\pi)^{-4} \int \mathrm{D}k \{\tfrac{1}{2} \tilde{I}^{\lambda\mu} L_\xi \tilde{G}_{\lambda\mu} + \tilde{m}^\lambda L_\xi \tilde{\Phi}_\lambda\}. \tag{8.50}$$

In (8.50), we have used (3.28) to bring L_ξ outside the Fourier transformation. Equation (8.49) will form the basis of the revised definition of the moments, while (8.50) decomposes into the two equations of motion.

These are the most important equations of the theory. Before continuing, let us consider some of their features. We have already seen that the allowed fields $\xi^\alpha(z, x)$ include all Killing vectors that the space-time may possess. So suppose that $\xi^\alpha(x)$ is a Killing vector field satisfying

$$L_\xi F_{\alpha\beta} = 0 \quad \text{as well as} \quad L_\xi g_{\alpha\beta} = 0. \tag{8.51}$$

We may then use it in (8.50), giving a ξ^α independent of z, if we take $A_\kappa = \xi_\kappa$ and $B_{\kappa\lambda} = \nabla_{[\kappa} \xi_{\lambda]}$. However, (8.51) implies also that

$$L_\xi \tilde{G}_{\kappa\lambda} = 0 \quad \text{and} \quad L_\xi \tilde{\Phi}_\lambda = 0, \tag{8.52}$$

so that the right hand side of (8.50) vanishes. We thus recover equation (I, 5.5). But whereas in I we could only derive those components of the equations of motion corresponding to conserved quantities, we now see quantitatively how forces are produced by a breakdown of the symmetry conditions (8.51).

Consider now equation (8.49). The concept of 'reduced moments' relies on the use of the differential relations (1.31) and (1.32) to eliminate redundant components from the complete moments. It is thus inherent in this concept that the gravitational and electromagnetic forces described by (1.32) will occur in the definitions of the reduced moments of $T^{\alpha\beta}$ even though they do not do so in the complete moments. For this reason we do not expect the gravitational analogue of (7.20) to be as simple as that equation; somehow the applied forces must enter. We see in (8.49) how they occur; they are responsible for the Λ-terms. We show now that these terms are remarkable in their simplicity. At a particular point $z(s)$, set up a normal coordinate system as in the discussion following equation (3.16). Then from that discussion and equations (8.40) and (8.20a), we see that the coefficients of Λ^κ in (8.49) are simply the values in normal coordinates of

$$F_{\alpha\beta} \quad \text{and} \quad [\alpha\beta,\gamma] := \tfrac{1}{2}\partial_{\{\alpha}g_{\gamma\beta\}}.$$

The first is simply the electromagnetic field tensor. The second is a Christoffel symbol of the first kind, which is about the most naive and basic description of the strength of the gravitational field. About the simplest way of giving a covariant meaning to a non-tensorial quantity such as $[\alpha\beta,\gamma]$ is to evaluate it in normal coordinates. The coefficients of Λ^κ are thus about the simplest conceivable measures of the electromagnetic and gravitational fields. This simplicity is regarded by the author as ample justification for taking (8.49) as defining the reduced moments, despite the fact that the explicit integral expressions for the moments to be given in the next section are quite complicated.

It should be remarked that it is possible to proceed also with the alternative (8.20b), which we later abandoned. This yields simpler explicit integrals for the moments, at the expense of substantially complicating the analogue of (8.49). It is this choice that puts the equations of motion in the form given, without proof, in Dixon (1973). At the time of writing that paper, I was unaware of the result (8.49) arising from the choice (8.20a), and I now consider that this considerably outweighs the extra complication of the explicit forms for the moments.

9. Explicit evaluation of the moments

The next task is to fill in the logical gap left after equation (8.16), by proving that moments do exist satisfying this equation under the stated conditions. We do this by obtaining explicit expressions for the moments as integrals of $T^{\alpha\beta}$ over the hypersurfaces $\Sigma(s)$. This involves expressing $h_{\alpha\beta\gamma\delta}$, ψ_α and χ in terms of $\tilde{H}_{\kappa\lambda\mu\nu}$, which we do successively. In this section we shall let $S(s)$, S and $\tilde{S}$ have the values given in (7.9).

From (2.25) and the Fourier inversion theorem, we have

$$H_{\kappa\lambda\mu\nu}(z,X) = (2\pi)^{-4}\int_{T_z}\tilde{H}_{\kappa\lambda\mu\nu}(z,k)\exp(-\mathrm{i}k.X)\,\mathrm{D}k. \tag{9.1}$$

But also, by (3.15),

$$h_{\alpha\beta\gamma\delta} = \sigma_{\dot{\alpha}}{}^{\kappa}\sigma_{\dot{\beta}}{}^{\lambda}\sigma_{\dot{\gamma}}{}^{\mu}\sigma_{\dot{\delta}}{}^{\nu}H_{\kappa\lambda\mu\nu} \tag{9.2}$$

in some neighbourhood of $S(s)$, since $h_{\alpha\beta\gamma\delta} = \mathrm{Exp}_A H_{\kappa\lambda\mu\nu}$ there. If we expand the exponential in (9.1) in a power series, we thus see that

$$\int_{\Sigma(s)} h_{\alpha\gamma\beta\delta}\sigma^{\gamma}\sigma^{\delta}\mathfrak{T}^{\alpha\beta}w^{\epsilon}\,\mathrm{d}\Sigma_{\epsilon} = (2\pi)^{-4}\int \mathrm{D}k\,\tilde{H}_{\lambda\mu\nu\rho}\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}k_{\kappa_1}\dots k_{\kappa_n}t^{\kappa_1\dots\kappa_n\lambda\nu\mu\rho}(s), \tag{9.3}$$

where the t's are defined by

$$t^{\kappa_1\dots\kappa_n\lambda\mu}(s) := (-1)^n\int_{\Sigma(s)} \sigma^{\kappa_1}\dots\sigma^{\kappa_n}\sigma^{\lambda}_{\ .\alpha}\sigma^{\mu}_{\ .\beta}\,\mathfrak{T}^{\alpha\beta}w^{\gamma}\,\mathrm{d}\Sigma_{\gamma} \quad \text{if} \quad n \geqslant 2. \tag{9.4}$$

To treat ψ_{α}, we must first define a propagator $\underset{n}{\Theta}^{\kappa\lambda\mu\nu}(z,y)$ for each $n \geqslant 2$, analogous to the $\underset{n}{\Theta}^{\kappa\lambda}$ defined by (II, 5.55). If $x(u)$ is any geodesic with affine parameter u, and if $x(0) = z$, $x(1) = y$, then we put

$$\underset{n}{\Theta}^{\kappa\lambda\mu\nu}(z,y) = (n-1)\int_0^1 \sigma^{\kappa\alpha}\sigma^{(\mu}_{\ .\alpha}\sigma^{\nu)}_{\ .\beta}\sigma^{\lambda\beta}u^{n-2}\,\mathrm{d}u \quad \text{for} \quad n \geqslant 2, \tag{9.5}$$

where the arguments of the σ's are $(z, x(u))$. In flat space we get

$$\underset{n}{\Theta}^{\kappa\lambda\mu\nu}(z,y) = g^{\kappa(\mu}g^{\nu)\lambda} \tag{9.6}$$

for all n and all y, which is why the factor $(n-1)$ is included in the definition. Now $\psi_{\alpha}(x)$ is defined, by (6.42) and (6.45), in terms of the solution of (6.38) when the functions $a_{\kappa}(s)$ and $b_{\kappa\lambda}(s)$ used in the initial conditions (6.48) are taken as zero. But under these conditions we showed that (6.38) can be integrated once to give the simpler equation (6.52). We thus obtain ψ from (6.52) with $e_{\alpha\beta}$ given by (8.15). If we then further use (9.1) and (9.2), the resulting equation can be explicitly integrated to give, for $x \in S(s)$,

$$\psi_{\alpha}(x) = -2(2\pi)^{-4}H_{\alpha\kappa}v_{\tau}\sigma^{\mu}\sigma^{\rho}\int \mathrm{D}k\,\tilde{H}_{\lambda\mu\nu\rho}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\mathrm{i}k.\sigma)^n\,\underset{n+2}{\Theta}^{\kappa\tau\lambda\nu}, \tag{9.7}$$

where $k.\sigma := k^{\lambda}\sigma_{\lambda}(z(s),x)$ and the arguments of all bitensors are $(z(s),x)$.

To evaluate $\chi(x)$, we must substitute (9.7) into (8.9). To integrate the result, we need yet another propagator, defined in the notation of (9.5) by

$$\underset{n}{\Phi}^{\kappa\lambda\mu\nu}(z,y) = (n+1)\int_0^1 F_{\alpha\beta}H^{\alpha\kappa}H^{\beta}_{\ .\rho}\,\underset{n}{\Theta}^{\rho\lambda\mu\nu}u^n\,\mathrm{d}u \quad \text{for} \quad n \geqslant 2. \tag{9.8}$$

We then find that for $x \in S(s)$,

$$\chi(x) = 2(2\pi)^{-4}v_{\tau}\sigma_{\kappa}\sigma^{\mu}\sigma^{\rho}\int \mathrm{D}k\,\tilde{H}_{\lambda\mu\nu\rho}\sum_{n=0}^{\infty}\frac{1}{(n+1)!\,(n+3)}(\mathrm{i}k.\sigma)^n\,\underset{n+2}{\Phi}^{\kappa\tau\lambda\nu}, \tag{9.9}$$

where again the arguments of all bitensors are $(z(s),x)$.

From (9.7) and (9.9) we see that

$$\int_{\Sigma(s)} [\chi\mathfrak{J}^{\beta} - \mathfrak{T}^{\alpha\beta}\psi_{\alpha}]\,\mathrm{d}\Sigma_{\beta} = (2\pi)^{-4}v_{\tau}\int \mathrm{D}k\,\tilde{H}_{\lambda\mu\nu\rho}\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{(n+1)!}k_{\kappa_1}\dots k_{\kappa_n}p^{\kappa_1\dots\kappa_n\lambda\nu\mu\rho\tau} \tag{9.10}$$

where, for $n \geqslant 2$,

$$p^{\kappa_1\dots\kappa_n\lambda\mu\nu} = 2(-1)^n\int_{\Sigma(s)} \sigma^{\kappa_1}\dots\sigma^{\kappa_n}\left[\underset{n}{\Theta}^{\rho\nu\lambda\mu}H_{\alpha\rho}\,\mathfrak{T}^{\alpha\beta} + \frac{1}{n+1}\underset{n}{\Phi}^{\rho\nu\lambda\mu}\sigma_{\rho}\mathfrak{J}^{\beta}\right]\mathrm{d}\Sigma_{\beta}. \tag{9.11}$$

Taken together with (9.3), we thus see that (8.16) holds under the required conditions if

$$J^{\kappa_1 \dots \kappa_n \lambda\mu\nu\rho} = t^{\kappa_1 \dots \kappa_n [\lambda[\nu\mu]\rho]} + \frac{1}{n+1} p^{\kappa_1 \dots \kappa_n [\lambda[\nu\mu]\rho]\tau} v_\tau \quad \text{for} \quad n \geqslant 0. \tag{9.12}$$

These are easily verified to satisfy the required symmetry and orthogonality conditions (5.33) and (5.36). It remains to check the behaviour of $\tilde{I}^{\kappa\lambda}$ for large k^λ, and the assumptions made concerning the support of the functional I. To do this, we must evaluate the moments $I^{\cdots}$ using (5.35) and then explicitly sum the series for $\tilde{I}^{\kappa\lambda}$ given in (4.16). The method of summing the series follows that used by Dixon (1967) in the derivation of equations (3.34) and (3.35) of that paper. The resulting lengthy expression has little intrinsic interest, and so it will not be given here but only the conclusions that can be drawn from it, as the method of arriving at them is well illustrated in the earlier paper. We can first deduce that $\tilde{I}^{\kappa\lambda}(s, k)$ diverges as $k \to \infty$ at most quadratically in k. As this guarantees the convergence of all the k-space integrations, we can then substitute this expression for $\tilde{I}^{\kappa\lambda}$ into the right hand side of (8.14). The k-space integration turns out to be easily evaluated using the Fourier inversion theorem even when $E_{\lambda\mu}$ is not required to satisfy (5.20), the result then being an integral expression involving $E_{\lambda\mu}$ itself. The support $\tilde{S}$ of the functional I can be read off from this expression, and is seen to be as given in (7.9), as required.

This completes the proof of the existence of the reduced moments as provisionally defined in §8. We have incidentally obtained explicit expressions for them, but these are somewhat unwieldy. Their importance lies mainly in their mere existence; for theoretical work with the reduced moments, one will generally use instead one of the defining relations (8.14) or (8.49). It is for this reason that the simplicity of (8.49) is considered to outweigh the complexity of the integral expressions, as remarked in the previous section.

10. Statement of the main theorem

We can at last state and prove the main result of this paper, namely the uniqueness theorem for the reduced moments of $T^{\alpha\beta}$. Since the defining relation (8.49) involves the moments of J^α as well as those of $T^{\alpha\beta}$, we shall state the result as a combined theorem for both fields, although the part referring to J^α has already been proved in §7. The theorem will be stated in as self-contained a form as possible without undue repetition of lengthy definitions and conditions, and clear reference will be made to any previous statements when required. In this way the statement of the theorem can also act as a summary of the work of the paper so far.

Consider a space-time manifold M containing an electromagnetic field described by an anti-symmetric tensor $F_{\alpha\beta}$ satisfying

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0. \tag{10.1}$$

Suppose that this space-time contains a charged body of finite extent, occupying a world tube W. The theorem connects two alternative descriptions of such a body, provided that it is not too large. For simplicity we also assume that the body is convex. These restrictions are made precise by requiring W to satisfy the conditions given in the fourth paragraph of §7.

The first description is in terms of a charge-current vector J^α and a symmetric energy-momentum tensor $T^{\alpha\beta}$. These are required to be of class C^1, i.e. to have continuous first derivatives. For convenience, we shall also suppose that J^α and $T^{\alpha\beta}$ are both nonzero throughout W, so that they each have support W.

The second description is by two infinite sets of multipole moments, given as tensor fields along a C^∞ timelike world line l. We parametrize l as $z^\lambda(s)$, where s is not necessarily the proper time, and write the moments as $m^{\lambda_1\ldots\lambda_n\mu}(s)$ and $I^{\lambda_1\ldots\lambda_n\mu\nu}(s)$. They are defined for each $n \geqslant 0$, and are required to satisfy the symmetry conditions

$$\left.\begin{aligned} m^{\lambda_1\ldots\lambda_n\mu} &= m^{(\lambda_1\ldots\lambda_n)\mu} \quad \text{for} \quad n \geqslant 2 \\ \text{and} \qquad m^{(\lambda_1\ldots\lambda_n\mu)} &= 0 \quad \text{for} \quad n \geqslant 1, \end{aligned}\right\} \tag{10.2}$$

and

$$\left.\begin{aligned} I^{\lambda_1\ldots\lambda_n\mu\nu} &= I^{(\lambda_1\ldots\lambda_n)(\mu\nu)} \quad \text{for} \quad n \geqslant 0, \\ I^{(\lambda\mu\nu)} = 0, \quad \text{and} \quad I^{(\lambda_1\ldots\lambda_n\mu)\nu} &= 0 \quad \text{for} \quad n \geqslant 2. \end{aligned}\right\} \tag{10.3}$$

But l alone is not a sufficient reference frame to which to attach the moments. In addition, we choose a C^∞ field of timelike unit vectors n^λ along l. The moments are then required to satisfy the orthogonality conditions

$$\left.\begin{aligned} n_{\lambda_1} m^{\lambda_1\ldots\lambda_{n-1}[\lambda_n\mu]} &= 0 \quad \text{for} \quad n \geqslant 2 \\ \text{and} \qquad n_{\lambda_1} I^{\lambda_1\ldots\lambda_{n-2}[\lambda_{n-1}[\lambda_n\mu]\nu]} &= 0 \quad \text{for} \quad n \geqslant 3. \end{aligned}\right\} \tag{10.4}$$

In the second of equations (10.4), the antisymmetrization denoted by the square brackets is over (λ_{n-1}, μ) and (λ_n, ν) independently, in accordance with the notation of Schouten (1954). The choice of l and n^λ is fairly free, but we require them to satisfy certain geometrical restrictions which are again given in the fourth paragraph of §7.

The two descriptions are linked through the use of the moment generating functions $\tilde{m}^\lambda$ and $\tilde{I}^{\lambda\mu}$, defined by

$$\tilde{m}^\lambda(s,k) := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\kappa_1} \ldots k_{\kappa_n} m^{\kappa_1\ldots\kappa_n\lambda}(s) \tag{10.5}$$

and

$$\tilde{I}^{\lambda\mu}(s,k) := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\kappa_1} \ldots k_{\kappa_n} I^{\kappa_1\ldots\kappa_n\lambda\mu}(s), \tag{10.6}$$

where k^λ is a vector at $z^\lambda(s)$. As a further condition on the moments, we suppose that for each fixed s, these functions diverge as $k \to \infty$ no faster than a polynomial in k. We see that $\tilde{m}^\lambda$ and $\tilde{I}^{\lambda\mu}$ are not true tensor fields on M; their domain of definition is really that part of the tangent bundle TM of M over l. They are such that their value at $X \in T_z(M)$ is a tensor on M at z. As discussed in §2, a field with this property is said to be a tensor field on TM over τ, where $\tau: TM \to M$ is the projection map of the tangent bundle.

If E (indices suppressed) is such a tensor field, its Fourier transform $\tilde{E}$ is defined by

$$\tilde{E}(z,k) := \int_{T_z} E(z,X) \exp(\mathrm{i}k^\lambda X_\lambda)\, \mathrm{D}X, \tag{10.7}$$

where $\mathrm{D}X$ is the scalar volume element on T_z defined by (2.18). This is again a tensor field on TM over τ. Using this, we now define two functionals m and I. Their domains of definition are the spaces of all tensor fields on TM over τ, of the appropriate type, which are of class C^∞ and of compact support. They are given by

$$m[E_\lambda] := (2\pi)^{-4} \int \mathrm{d}s \int_{T_{z(s)}} \tilde{m}^\lambda(s,k)\, \tilde{E}_\lambda(z(s),k)\, \mathrm{D}k \tag{10.8}$$

and

$$I[E_{\lambda\mu}] := (2\pi)^{-4} \int \mathrm{d}s \int_{T_{z(s)}} \tilde{I}^{\lambda\mu}(s,k)\, \tilde{E}_{\lambda\mu}(z(s),k)\, \mathrm{D}k, \tag{10.9}$$

where $E_{\lambda\mu} = E_{(\lambda\mu)}$. The above assumption on the behaviour of $\tilde{m}^{\lambda}$ and $\tilde{I}^{\lambda\mu}$ ensures that these integrals are absolutely convergent.

We next define two functionals associated with J^{α} and $T^{\alpha\beta}$, but these are much simpler. If ϕ_{α} and $\phi_{\alpha\beta}$ are C^{∞} tensor fields on M of compact support, and $\phi_{\alpha\beta} = \phi_{(\alpha\beta)}$, we put

$$\langle J^{\alpha}, \phi_{\alpha}\rangle := \int J^{\alpha}\phi_{\alpha}\sqrt{(-g)}\,\mathrm{d}^4x \tag{10.10}$$

and

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta}\rangle := \int T^{\alpha\beta}\phi_{\alpha\beta}\sqrt{(-g)}\,\mathrm{d}^4x. \tag{10.11}$$

The linkage is then completed by expressing (10.10) and (10.11) in terms of m and I, but before doing so, we must first define an auxiliary bitensor field $\lambda^{\alpha}(z,x)$. This has scalar character at z and vector character at x, and such a λ^{α} is to be associated with every $\phi_{\alpha\beta}$. It satisfies two conditions. First, that

$$\lambda_{\alpha} \to 0, \quad \nabla_{\alpha}\lambda_{\beta} - \phi_{\alpha\beta} \to 0 \quad \text{as} \quad x \to z. \tag{10.12}$$

Secondly, for each $z \in W$, if $x(u)$ is a spacelike geodesic through z, then $\lambda^{\alpha}(z, x(u))$ satisfies

$$\frac{\delta^2}{\mathrm{d}u^2}\lambda_{\alpha} + R_{\alpha\beta\gamma\delta}\dot{x}^{\beta}\dot{x}^{\gamma}\lambda^{\delta} = \dot{x}^{\beta}\dot{x}^{\gamma}\nabla_{\{\beta}\phi_{\alpha\gamma\}} \tag{10.13}$$

along that portion of this geodesic lying within W. Here, $\dot{x}^{\alpha} := \mathrm{d}x^{\alpha}/\mathrm{d}u$, and the notation $\{\}$ around the indices is defined in appendix 1. These conditions determine $\lambda^{\alpha}(z,x)$ completely for all pairs of points $z, x \in W$ which have spacelike separation. We continue λ^{α} arbitrarily to all other points, subject only to its having class C^{∞}.

Two further definitions are needed before we can state the main hypothesis of the theorem. Let N be the region of TM on which the exponential map is well behaved. (This is defined more precisely in §3.) If $\phi^{\alpha\ldots}{}_{\beta\ldots}(z,x)$ is a bitensor field on M with scalar character at z, and if $\Phi^{\kappa\ldots}{}_{\lambda\ldots}(z,X)$ is a tensor field on TM over τ, we say that $\Phi = \mathrm{Exp}^{\Delta}\phi$ in a region $R \subset N$ if

$$\Phi^{\kappa\ldots}{}_{\lambda\ldots}(z,X) = (-\sigma_{\alpha}{}^{;\kappa})\ldots H^{\beta}{}_{.\lambda}\ldots\phi^{\alpha\ldots}{}_{\beta\ldots}(z,x), \tag{10.14}$$

where $x = \mathrm{Exp}\,X$, for all $(z,X) \in R$. The inverse relation is written as $\phi = \mathrm{Exp}_{\Delta}\,\Phi$. Here,

$$\sigma_{\alpha\kappa} = \partial^2\sigma/\partial x^{\alpha}\partial z^{\kappa},$$

where σ is the world function biscalar, and $H^{\alpha\kappa}$ is the matrix inverse of $(-\sigma_{\alpha\kappa})$. A simple interpretation of this using normal coordinates is given in §3. Also, for any tensor field on TM over τ, we denote partial differentiation with respect to the components of the argument vector X^{λ} by $\nabla_{*\lambda}$, as in (2.16).

Now let S' be a bounded closed subset of TM satisfying $S' \subset N$. Suppose that the functionals (10.10) and (10.11) satisfy

$$\langle J^{\alpha}, \phi_{\alpha}\rangle = m[\Phi_{\lambda}] \tag{10.15}$$

and

$$\langle T^{\alpha\beta}, \phi_{\alpha\beta}\rangle = I[\Phi_{\lambda\mu} + \tfrac{1}{2}\Lambda^{\kappa}\nabla_{*\{\lambda}G_{\kappa\mu\}}] + m[\Lambda^{\kappa}f_{\kappa\lambda}] \tag{10.16}$$

whenever Φ_{λ}, $\Phi_{\lambda\mu}$, $G_{\lambda\mu}$, $f_{\lambda\mu}$ and Λ^{κ} are all C^{∞} tensor fields on TM over τ, of compact support, which satisfy

$$\left.\begin{aligned} &\Phi_{\lambda} = \mathrm{Exp}^{\Delta}\phi_{\alpha}, \quad \Phi_{\lambda\mu} = \mathrm{Exp}^{\Delta}\phi_{\alpha\beta}, \quad G_{\lambda\mu} = \mathrm{Exp}^{\Delta}g_{\alpha\beta}, \\ &f_{\lambda\mu} = \mathrm{Exp}^{\Delta}F_{\alpha\beta} \quad \text{and} \quad \Lambda^{\kappa} = \mathrm{Exp}^{\Delta}\lambda^{\alpha} \end{aligned}\right\} \tag{10.17}$$

in some open neighbourhood of S'. Suppose further that no closed subset of S' also has this property. Then

(i) It is necessarily true that J^α and $T^{\alpha\beta}$ satisfy

$$\nabla_\alpha J^\alpha = 0 \tag{10.18}$$

and

$$\nabla_\beta T^{\alpha\beta} = -F^{\alpha\beta} J_\beta. \tag{10.19}$$

Conversely, if these equations are satisfied, then moments $m^{\cdots}$ and $I^{\cdots}$ and a region S' do exist satisfying all the conditions of the theorem.

(ii) The region S' is unique and is given by

$$S' = \bigcup_s \mathrm{Exp}^{-1}_{z(s)} (\Sigma(s) \cap W), \tag{10.20}$$

where $\Sigma(s)$ is the hypersurface formed by all geodesics through $z(s)$ orthogonal to $n^\lambda(s)$.

(iii) The moments are all uniquely determined, and for a given value of s they depend on J^α and $T^{\alpha\beta}$ only through their restrictions to $\Sigma(s) \cap W$.

(iv) The lowest three moments have the special forms

$$m^\lambda = qv^\lambda, \quad I^{\lambda\mu} = p^{(\lambda}v^{\mu)}, \quad I^{\kappa\lambda\mu} = S^{\kappa(\lambda}v^{\mu)}, \tag{10.21}$$

where $S^{\kappa\lambda} = -S^{\lambda\kappa}$ and $v^\lambda = \mathrm{d}z^\lambda/\mathrm{d}s$.

(v) The s-dependence of these lowest moments is uniquely determined. We have

$$\mathrm{d}q/\mathrm{d}s = 0, \tag{10.22}$$

and $\delta p^\kappa/\mathrm{d}s$ and $\delta S^{\kappa\lambda}/\mathrm{d}s$ are determined by (8.50).

(vi) q is the total charge of the body, while p^κ and $S^{\kappa\lambda}$ agree with the definitions of total momentum and angular momentum deduced in I on other grounds.

(vii) There are no other algebraic or differential relations between the moments, as a consequence of the 'conservation equations' (10.18) and (10.19), other than those given in (v).

In connexion with (iii), explicit expressions for the m's were given in II, while explicit expressions for the I's were derived in §9. The equations of motion referred to in (v) are considered in more detail in §§12 and 13 below.

11. The uniqueness proof

We now turn to the proof of those parts of the above theorem that have not yet been demonstrated. The converse part of (i) has already been proved, and result (ii) is shown by the same method as used in §7 for the corresponding result for J^α alone. We can thus again take $S' = \tilde{S}$, and let $\tilde{S}$, S and $S(s)$ be unambiguously given by (7.9). Note, incidentally, that (ii) follows either from (10.15) or (10.16), so that it is not invalidated when we consider the case $F_{\alpha\beta} = 0$, when all reference to J^α can be deleted from the theorem. So, as for J^α, we turn to the proof of (iii). The uniqueness of the m's has already been shown; this leaves that of the I's still to be proved. We do this by deducing both the provisional definition (8.14), and also (8.31) with (8.12) and (8.13), from (10.16). The method follows that used in §7 for J^α, but has additional complications due to the A-terms in (10.16).

As in (8.14), let $E_{\lambda\mu}$ be an arbitrary symmetric tensorial test field on TM over τ satisfying

$$X^\lambda E_{\lambda\mu} = 0. \tag{11.1}$$

Choose a symmetric test function $e_{\alpha\beta}(s, x)$ which satisfies

$$\sigma^\alpha e_{\alpha\beta} = 0 \tag{11.2}$$

identically, and

$$E_{\lambda\mu} = \mathrm{Exp}^A e_{\alpha\beta} \tag{11.3}$$

in some neighbourhood of $\tilde{S}$. From it, construct the functions $c_{\alpha\beta}(s,x)$ and $\omega_\alpha(s,x)$ of (6.2), using the method which starts in the paragraph containing (6.38). In so doing, the functions $\nu_\alpha(x)$ of (6.39) and $a_\kappa(s)$ and $b_{\kappa\lambda} = b_{[\kappa\lambda]}(s)$ of (6.48) will be chosen arbitrarily. From these, we then construct $\phi_{\alpha\beta}(x)$ by (6.43), $\psi_\alpha(x)$ by (6.45) with (6.42), $A_\kappa(s)$ and $B_{\kappa\lambda}(s)$ by (6.56) and $\xi^\alpha(s,x)$ by (6.70).

We shall apply (10.16) with this choice of $\phi_{\alpha\beta}$, but we must first show that we can modify the defining relations for λ^α without affecting the validity of (10.16). This is done in two stages. From (5.6), (5.7) and (10.17) we first note that the right hand side of (10.16) depends on the values taken by $\lambda^\alpha(z,x)$ only for $z \in l$. We can thus replace $\lambda^\alpha(z,x)$ by a more restrictive function $\lambda^\alpha(s,x)$ corresponding to $z = z(s)$. We next show that if we alter (10.13) by replacing $\phi_{\alpha\beta}(x)$ on the right hand side by $c_{\alpha\beta}(s,x)$, we do not affect the value of the right hand side of (10.16). From (5.6) and (5.7), we see that this is the case provided that the alteration does not affect the values taken by

$$L_\lambda, \quad \nabla_{*\mu}(X^\lambda L_\lambda), \quad L_{\kappa\lambda}, \quad \nabla_{*\mu}(X^\lambda L_{\kappa\lambda}) \quad \text{and} \quad \nabla_{*\mu\nu}(X^\kappa X^\lambda L_{\kappa\lambda}) \tag{11.4}$$

on $\tilde{S}$, where

$$L_{\kappa\lambda} := \Lambda^\mu \nabla_{*\{\kappa} G_{\mu\lambda\}}, \quad L_\lambda := \Lambda^\mu f_{\lambda\mu}. \tag{11.5}$$

Now λ^α is a special case of the ω^α of § 6, corresponding to taking $A_\kappa = 0$, $B_{\kappa\lambda} = 0$ in (6.55). The result (6.63) is thus applicable, and it shows that the replacement concerned does not affect the values of λ_α and $\nabla_\alpha \lambda_\beta$ for $x \in S(s)$. This is sufficient to show that the first four of the five expressions in (11.4) are unchanged on $\tilde{S}$. It remains to consider the final expression. We deal with this by showing that $X^\kappa X^\lambda L_{\kappa\lambda}$ is identically zero for all Λ^μ, so that this term is certainly unaltered.

Since $X^\kappa = -\operatorname{Exp}^\Delta \sigma^\alpha$, we get from (10.17) that around $\tilde{S}$,

$$X^\kappa G_{\kappa\lambda} = X_\lambda = X^\kappa g_{\kappa\lambda}. \tag{11.6}$$

By differentiation, this gives

$$X^\kappa \nabla_{*\mu} G_{\kappa\lambda} = g_{\mu\lambda} - G_{\mu\lambda}. \tag{11.7}$$

With (11.6) this gives

$$X^\kappa X^\lambda \nabla_{*\mu} G_{\kappa\lambda} = X^\kappa X^\lambda \nabla_{*\kappa} G_{\mu\lambda} = 0, \tag{11.8}$$

and hence with (11.5),

$$X^\kappa X^\lambda L_{\kappa\lambda} = 0 \tag{11.9}$$

identically, as required. This completes the above proof. On checking the various definitions involved, we see that we can now take

$$\lambda^\alpha = \omega^\alpha - \xi^\alpha. \tag{11.10}$$

With this choice of λ^α, and with $\phi_{\alpha\beta}$ constructed as specified above, choose tensor fields $\Phi_{\lambda\mu}$, $G_{\lambda\mu}$, $f_{\lambda\mu}$ and Λ^κ on TM over τ in accordance with (10.17). Now observe that (10.17) and (11.3) agree with (8.20*a*) and (8.21), and that (11.10) agrees with (6.69). It thus follows, as in § 8, that if $C_{\lambda\mu}$ is defined by (8.22*a*), then $C_{\lambda\mu} = \operatorname{Exp}^\Delta c_{\alpha\beta}$ in some neighbourhood of $\tilde{S}$. Together with (5.39), (6.6) and (6.43), this gives

$$I[\Phi_{\lambda\mu}] = I[C_{\lambda\mu}]. \tag{11.11}$$

But for the same reasons, (8.28) holds if M_λ is defined by (8.24). On using these together with (5.5) and the identity (8.25), we then obtain

$$I[\Phi_{\lambda\mu} + \tfrac{1}{2}\Lambda^\kappa \nabla_{*\{\lambda} G_{\kappa\mu\}}] = I[E_{\lambda\mu} + \tfrac{1}{2}L_\xi G_{\lambda\mu}] + \int \mathrm{d}s[I^{\lambda\mu}\nabla_\lambda \lambda_\mu + I^{\kappa\lambda\mu}\nabla_{\kappa\lambda}\lambda_\mu]. \tag{11.12}$$

The final step in treating this term of (10.16) is to substitute for λ^α from (11.10). To do so, we need to evaluate the coincidence limits $\langle \nabla_{(\alpha}\xi_{\beta)} \rangle$ and $\langle \nabla_{\alpha(\beta}\xi_{\gamma)} \rangle$, using the diamond bracket notation as in (8.46) and (8.47). This may be done from (6.70) using the results of the appendix of II, and we find that both are zero. Hence (11.12) becomes

$$I[\Phi_{\lambda\mu} + \tfrac{1}{2}\Lambda^\kappa \nabla_{*\{\lambda} G_{\kappa\mu\}}] = I[E_{\lambda\mu} + \tfrac{1}{2}L_\xi G_{\lambda\mu}] + \int \mathrm{d}s[I^{\lambda\mu}\nabla_\lambda \omega_\mu + I^{\kappa\lambda\mu}\nabla_{\kappa\lambda}\omega_\mu]. \tag{11.13}$$

We turn next to the electromagnetic terms in (10.16). From (10.17) and (8.40) we see that we can take

$$f_{\lambda\mu} = -2\nabla_{*[\lambda}\Phi_{\mu]}, \tag{11.14}$$

where Φ_μ is defined by (8.8). Also, if we define e_α and E_λ as in (8.5) and the following sentence, the steps leading from (8.33) to (8.48) all remain valid. We thus get from (8.48) and (6.70) that

$$m[A^\kappa f_{\kappa\lambda}] = m[E_\lambda + L_\xi \Phi_\lambda] - \int \mathrm{d}s\, qv^\lambda A^\kappa F_{\kappa\lambda}. \tag{11.15}$$

To continue, we must first note that (8.2) remains true, with a slight modification, with the present definitions and with $\bar{p}^\kappa$ and $\bar{S}^{\kappa\lambda}$ still defined by (8.3), even though (8.1) is now replaced by the weaker condition (6.43). The modification is the replacement of the final term by

$$-\int \nu_\alpha \nabla_\beta \mathfrak{T}^{\alpha\beta}\,\mathrm{d}^4x,$$

needed since we are no longer assuming (5.11). Similarly, the steps from (8.5) to (8.10) still hold, using the definitions there given. We can thus substitute for the three terms of (10.16) from the modified (8.2), and from (11.13) and (11.15). Do so, and then substitute for $m[E_\lambda]$ in the result from (8.6), and for ψ from (8.10). If we define $p^\kappa(s)$ and $S^{\kappa\lambda}(s)$ as in (8.12) and (8.13), the final result can be put in the form

$$\int \mathrm{d}s \int (e_{\alpha\beta}\mathfrak{T}^{\alpha\beta}w^\gamma - \psi_\alpha \mathfrak{T}^{\alpha\gamma} + \chi\mathfrak{J}^\gamma)\,\mathrm{d}\Sigma_\gamma = I[E_{\lambda\mu} + \tfrac{1}{2}L_\xi G_{\lambda\mu}] + m[L_\xi \Phi_\lambda] + \int \mathrm{d}s[I^{\lambda\mu}\nabla_\lambda \omega_\mu$$
$$+ I^{\kappa\lambda\mu}\nabla_{\kappa\lambda}\omega_\mu + a_\kappa p^\kappa + \tfrac{1}{2}b_{\kappa\lambda}S^{\kappa\lambda} - qv^\lambda A^\kappa F_{\kappa\lambda}] + \int \nu_\alpha(\nabla_\beta \mathfrak{T}^{\alpha\beta} + F^{\alpha\beta}\mathfrak{J}_\beta)\,\mathrm{d}^4x. \tag{11.16}$$

The required results all follow from this on using the arbitrariness, for given $e_{\alpha\beta}$, of $a_\kappa(s)$, $b_{\kappa\lambda}(s)$ and $\nu_\alpha(x)$. The other variables occurring in (11.16) which depend on these arbitrary quantities are ξ^α, ω_α and A^κ, their dependence being given by (6.39), (6.56) and (6.70).

We first consider the special case $E_{\lambda\mu} = 0$, $e_{\alpha\beta} = 0$, $a_\kappa = 0$, $b_{\kappa\lambda} = 0$. We may then take $\omega_\alpha(s, x)$ as independent of s by (6.38) and (6.48), and hence by (6.39) we have $\omega_\alpha(s, x) = \nu_\alpha(x)$ for all s. This also implies $\psi_\alpha = 0$ by (6.45) and (6.42), and hence $\chi = 0$ by (8.9). For this case, (11.16) thus reduces to

$$I[\tfrac{1}{2}L_\xi G_{\lambda\mu}] + m[L_\xi \Phi_\lambda] + \int \mathrm{d}s[I^{\lambda\mu}\nabla_\lambda \nu_\mu + I^{\kappa\lambda\mu}\nabla_{\kappa\lambda}\nu_\mu - qv^\lambda A^\kappa F_{\kappa\lambda}] = -\int \nu_\alpha(\nabla_\beta \mathfrak{T}^{\alpha\beta} + F^{\alpha\beta}\mathfrak{J}_\beta)\,\mathrm{d}^4x, \tag{11.17}$$

where ξ^α is given by (6.69) and $\qquad A_\kappa = \nu_\kappa, \quad B_{\kappa\lambda} = \nabla_{[\kappa}\nu_{\lambda]}.$ (11.18)

The left hand side of (11.17) thus vanishes if $\nu_\alpha = 0$ in the neighbourhood of the world line l. Since $\nabla_\beta \mathfrak{T}^{\alpha\beta} + F^{\alpha\beta}\mathfrak{J}_\beta$ is continuous by hypothesis, (11.17) can thus hold for all ν_α only if both sides are identically zero. The vanishing of the right hand side gives (10.19), and completes the proof of result (i). We now consider the implications of the vanishing of the left hand side.

For this, it is convenient to use temporarily a coordinate system in which l is given by $z^k \overset{*}{=}$ constant, $k = 1, 2, 3$, and to let latin indices run only through the values 1, 2, 3. Since we can specify ν_α separately on each hypersurface $z^0 \overset{*}{=}$ constant, subject only to restrictions of continuity, we see that along l we can give independent values to ν_α and its spatial derivatives

$$\partial_k \nu_\alpha, \partial_{kl}\nu_\alpha, \ldots.$$

Consequently, while keeping ν_λ and $\nabla_\kappa \nu_\lambda$ fixed along l, we can still vary $\nabla_{(kl)}\nu_\alpha$ arbitrarily. Since second derivatives of ν_α occur in (11.17) only in the term involving $I^{\kappa\lambda\mu}\nabla_{\kappa\lambda}\nu_\mu$, this arbitrariness implies

$$I^{(kl)\mu} \overset{*}{=} 0. \tag{11.19}$$

But from (10.3),
$$2I^{(\lambda\mu)\nu}+I^{\nu\lambda\mu} = 0, \tag{11.20}$$
and hence (11.19) implies
$$I^{\mu kl} \overset{*}{=} 0. \tag{11.21}$$
Now by construction, $v^k \overset{*}{=} 0$, and hence (11.21) can be written in the covariant form
$$(A^\kappa_\nu - k^\kappa k_\nu)\,(A^\lambda_\rho - k^\lambda k_\rho)\,I^{\mu\nu\rho} = 0, \tag{11.22}$$
where k^λ is a unit vector in the direction of v^λ. Since $I^{\mu\kappa\lambda} = I^{\mu(\kappa\lambda)}$, on expansion of (11.22) we see that $I^{\mu\kappa\lambda}$ has the form
$$I^{\mu\kappa\lambda} = \hat{S}^{\mu(\kappa}v^{\lambda)} \tag{11.23}$$
for a suitable tensor field $\hat{S}^{\mu\kappa}(s)$. On substituting (11.23) into the second of the conditions (10.3), we see that $\hat{S}^{\mu\kappa}$ must be antisymmetric. This proves the part of (iv) referring to $I^{\kappa\lambda\mu}$.

Let us now briefly return to (11.16) itself, and use (11.23) to simplify the relevant term in this equation. Together with (6.55) and (6.60), we find that it gives
$$I^{\kappa\lambda\mu}\nabla_{\kappa\lambda}\,\omega_\mu = \tfrac{1}{2}\hat{S}^{\kappa\lambda}\left(\frac{\delta}{\mathrm{d}s}B_{\kappa\lambda} - b_{\kappa\lambda}\right) - \tfrac{1}{2}R_{\kappa\lambda\mu\nu}\hat{S}^{\kappa\lambda}v^\mu A^\nu. \tag{11.24}$$
In the special case of (11.17), this is applicable with $\omega_\mu = \nu_\mu$, $b_{\kappa\lambda} = 0$, and on using it we obtain the left hand side of (11.17) in a form which no longer involves second derivatives of ν_α.

The terms in (11.17) involving first derivatives of ν_α now divide into two groups, those only involving $B_{\kappa\lambda} = \nabla_{[\kappa}\nu_{\lambda]}$, and those only involving $\nabla_{(\kappa}\nu_{\lambda)}$. The second group consists only of the term in $I^{\lambda\mu}\nabla_\lambda\nu_\mu$. Now we saw above that along l we may specify ν_α and $\nabla_k\nu_\alpha$ arbitrarily. It is easily seen that this is equivalent to giving A_κ and $B_{\kappa\lambda}$ of (11.18), together with $\nabla_{(k}\nu_{l)}$, arbitrarily. The identical vanishing of the left hand side of (11.17) implies that the coefficient of $\nabla_{(k}\nu_{l)}$ vanishes, and hence
$$I^{kl} \overset{*}{=} 0. \tag{11.25}$$
On treating this similarly to (11.19), we find that there exists a vector field $\hat{p}^\lambda(s)$ such that
$$I^{\lambda\mu} = \hat{p}^{(\lambda}v^{\mu)}, \tag{11.26}$$
which completes the proof of result (iv). We now also use this in (11.16), by deducing from (11.26), (6.48) and (6.55) that
$$I^{\lambda\mu}\nabla_\lambda\,\omega_\mu = \hat{p}^\lambda(\delta A_\lambda/\mathrm{d}s - a_\lambda) + \hat{p}^{[\lambda}v^{\mu]}B_{\lambda\mu}. \tag{11.27}$$
This is again applicable in (11.17) with $\omega_\mu = \nu_\mu$, $a_\lambda = 0$. The resulting form of the left hand side of (11.17) involves ν_α only in the form of A_κ and $B_{\kappa\lambda}$, both of which are arbitrary. On integrating by parts the two terms involving s-derivatives of A_κ and $B_{\kappa\lambda}$, we then recover the equations of motion (8.50), except that p^κ and $S^{\kappa\lambda}$ are replaced by $\hat{p}^\kappa$ and $\hat{S}^{\kappa\lambda}$ respectively. This is the required result (v).

On using these above results, we can simplify (11.16) to the form
$$\int \mathrm{d}s \int (e_{\alpha\beta}\mathfrak{T}^{\alpha\beta}w^\gamma - \psi_\alpha\mathfrak{T}^{\alpha\gamma} + \chi\mathfrak{J}^\gamma)\,\mathrm{d}\Sigma_\gamma = I[E_{\lambda\mu}] + \int \mathrm{d}s\,[a_\kappa(p^\kappa - \hat{p}^\kappa) + \tfrac{1}{2}b_{\kappa\lambda}(S^{\kappa\lambda} - \hat{S}^{\kappa\lambda})]. \tag{11.28}$$
The arbitrary fields a_κ and $b_{\kappa\lambda}$ occur only in the final integral, which must thus vanish identically, giving
$$\hat{p}^\kappa = p^\kappa, \quad \hat{S}^{\kappa\lambda} = S^{\kappa\lambda}, \tag{11.29}$$
which is result (vi). The remainder of equation (11.28) then gives the integral over s of (8.14), from which (8.14) itself can be derived following the method used in §7 to similarly treat (7.29).

Since (8.14) was our provisional definition of the moments of $T^{\alpha\beta}$, this completes the proof of result (iii).

It remains only to prove (vii). This is equivalent to showing that we have extracted all the information contained in (11.16) and its electromagnetic analogue (7.28), so that these equations are identically satisfied. A scrutiny of the above calculations shows that this is indeed so. Alternatively a direct proof could be given, as was done for result (vi) of § 7. This completes the proof.

12. Equations of motion

We now study in more detail the equations of motion given by (8.50), so as to obtain separate equations for $\delta p^\lambda/\mathrm{d}s$ and $\delta S^{\lambda\mu}/\mathrm{d}s$. We first show that its right hand side has vanishing contributions from $I^{\kappa\lambda}$, $I^{\kappa\lambda\mu}$ and m^λ, so that p^κ, $S^{\kappa\lambda}$ and q appear only on the left hand side of this equation. On using (II, 7.2), we find that the contribution from these moments to this right hand side is

$$\tfrac{1}{2}I^{\kappa\lambda}\langle L_\xi G_{\kappa\lambda}\rangle + \tfrac{1}{2}I^{\kappa\lambda\mu}\delta^\alpha_\mu\langle\nabla_\alpha L_\xi G_{\kappa\lambda}\rangle + m^\lambda\langle L_\xi \Phi_\lambda\rangle. \tag{12.1}$$

The third term of this vanishes by (8.42) and (8.46). Each of the other two terms can be seen to also vanish, by the same method, on using

$$\langle G_{\kappa\lambda}\rangle = g_{\kappa\lambda}, \quad \langle\nabla_\alpha G_{\kappa\lambda}\rangle = 0 \quad \text{and} \quad \langle\nabla_\mu G_{\kappa\lambda}\rangle = 0, \tag{12.2}$$

which follow from (10.14), (10.17) and (II, A 18).

Let us now write

$$\hat{I}^{\lambda\mu} := \sum_{n=2}^{\infty}\frac{(-\mathrm{i})^n}{n!}k_{\kappa_1}\dots k_{\kappa_n}I^{\kappa_1\dots\kappa_n\lambda\mu} \tag{12.3}$$

and

$$\hat{m}^\lambda := \sum_{n=1}^{\infty}\frac{(-\mathrm{i})^n}{n!}k_{\kappa_1}\dots k_{\kappa_n}m^{\kappa_1\dots\kappa_n\lambda}, \tag{12.4}$$

which differ from $\tilde{I}^{\lambda\mu}$ and $\tilde{m}^\lambda$ only by the omission of the first two and one terms respectively. Then the above calculation shows that we can replace $\tilde{I}^{\lambda\mu}$ and $\tilde{m}^\lambda$ in (8.50) by $\hat{I}^{\lambda\mu}$ and $\hat{m}^\lambda$ respectively. Note also that (4.17) and (4.18) imply the identities

$$k^\lambda\hat{m}_\lambda = 0, \quad k^\lambda\hat{I}_{\lambda\mu} = 0, \tag{12.5}$$

from which we can deduce by differentiation that

$$\hat{I}_{\kappa\mu} = -k^\lambda\nabla_{*\mu}\hat{I}_{\kappa\lambda}, \quad \hat{m}_\mu = -k^\lambda\nabla_{*\mu}\hat{m}_\lambda. \tag{12.6}$$

Since equations (3.26) and (6.70) agree, we can use (3.27) to evaluate the Lie derivatives in (8.50). The coefficients of the arbitrary fields A_κ and $B_{\kappa\lambda}$ can then be separately equated to zero to give the two desired equations of motion. After an integration by parts and the use of (12.6), the resulting equations can be put in the form

$$\delta p_\kappa/\mathrm{d}s = \tfrac{1}{2}v^\lambda S^{\mu\nu}R_{\kappa\lambda\mu\nu} - qv^\lambda F_{\kappa\lambda} + (2\pi)^{-4}\int \mathrm{D}k\,\{\tfrac{1}{2}\hat{I}^{\lambda\mu}\nabla_{\kappa*}\tilde{G}_{\lambda\mu} + \hat{m}^\lambda\nabla_{\kappa*}\tilde{\Phi}_\lambda\} \tag{12.7}$$

and

$$\delta S_{\kappa\lambda}/\mathrm{d}s = 2p_{[\kappa}v_{\lambda]} + 2(2\pi)^{-4}\mathrm{i}\int \mathrm{D}k\,\{\tilde{G}_{\mu\nu[\kappa}\nabla_{*\lambda]}\hat{I}^{\mu\nu} - \tilde{\Phi}_{\mu[\kappa}\nabla_{*\lambda]}\hat{m}^\mu\}, \tag{12.8}$$

where

$$\Phi_{\kappa\lambda} := 2\nabla_{*[\lambda}\Phi_{\kappa]}, \quad G_{\kappa\lambda\mu} := \tfrac{1}{2}\nabla_{*\{\kappa}G_{\mu\lambda\}}. \tag{12.9}$$

The new fields defined in (12.9) are just the coefficients of A^κ in (8.49). They have a simple interpretation, which was discussed in the paragraph following equation (8.52). Note that $\Phi_{\kappa\lambda}$ is the $f_{\kappa\lambda}$ of (10.17); the notation $\Phi_{\kappa\lambda}$ seems more coherent, but it could not be used in § 10 as $\Phi_{\kappa\lambda}$ there had a different meaning.

An equivalent but slightly simpler form can be given to these equations with the use of generalized functions. Although the inverse Fourier transforms of $\hat{I}^{\lambda\mu}$ and $\hat{m}^{\lambda}$ do not exist as ordinary functions, they do exist as generalized functions (distributions). They are real, and have compact support on each tangent space $T_{z(s)}$. It is actually more convenient to use the inverse Fourier transforms of their complex conjugates rather than of these functions themselves. We shall denote them by $\hat{T}^{\lambda\mu}(s, X)$ and $\hat{J}^{\lambda}(s, X)$ respectively, due to their close connexion with the corresponding tensor fields $T^{\alpha\beta}(x)$ and $J^{\alpha}(x)$. Then on using (2.27) and (II, 2.6), we can write (12.7) and (12.8) as

$$\delta p_{\kappa}/\mathrm{d}s = \tfrac{1}{2}v^{\lambda}S^{\mu\nu}R_{\kappa\lambda\mu\nu} - qv^{\lambda}F_{\kappa\lambda} + \int \mathrm{D}X[\tfrac{1}{2}\hat{T}^{\lambda\mu}\nabla_{\kappa *}G_{\lambda\mu} + \hat{J}^{\lambda}\nabla_{\kappa *}\Phi_{\lambda}] \tag{12.10}$$

and

$$\delta S_{\kappa\lambda}/\mathrm{d}s = 2p_{[\kappa}v_{\lambda]} + 2\int \mathrm{D}X[\hat{J}^{\mu}\Phi_{\mu[\lambda}X_{\kappa]} - \hat{T}^{\mu\nu}G_{\mu\nu[\lambda}X_{\kappa]}]. \tag{12.11}$$

In this form the expression for the couple, given by the integral in (12.11), shows up as the moment of an 'apparent force density'

$$\hat{J}^{\mu}\Phi_{\mu\lambda} - \hat{T}^{\mu\nu}G_{\mu\nu\lambda}. \tag{12.12}$$

When viewed in normal coordinates about z, as discussed in § 8, this bears a remarkable resemblance to the right hand side of the form

$$g_{\alpha\gamma}\partial_{\beta}\mathfrak{T}^{\gamma\beta} = \mathfrak{J}^{\beta}F_{\beta\alpha} - \mathfrak{T}^{\beta\gamma}[\beta\gamma, \alpha] \tag{12.13}$$

of the conservation law (1.32). Note, however, that the 'apparent force density' given by the integrand in (12.10) differs from (12.12). This is perhaps to be expected, since the corresponding 'apparent momentum densities' used in the equations (8.12) and (8.13) which define p^{κ} and $S^{\kappa\lambda}$ also differ.

13. Multipole approximations

The forms of the equations of motion given in the preceding section are exact. Consequently, they are valid even for 'massive bodies', when the fields $g_{\alpha\beta}$ and $F_{\alpha\beta}$ include self-contributions from the body whose motion is being investigated. But they are rather difficult to handle, since they involve an integration whose integrand contains an infinite series in the multipole moments of the body. Now if the external field, i.e. that due to all bodies other than the one under consideration, varies sufficiently gradually over a spacelike section of the body, we might expect that the contributions from the higher order moments would be negligible. This speculation is based on the corresponding Newtonian result, in which theory the separation of the external field from the self-field is trivial, and the vanishing of the resultant self-force and self-couple is easily proved. In general relativity, however, there is no natural way of separating the external and self-fields. As a result, the above speculation is difficult even to formulate in more precise terms unless the self-field of the body is negligible. In this case the body is called a 'test body'.

In the present section we shall restrict ourselves to test bodies. The author hopes to return to the case of a massive body in a subsequent paper. The external fields are then the full fields appearing in (12.7) and (12.8). We shall thus suppose that $G_{\kappa\lambda}$ and Φ_{κ} vary slowly throughout a neighbourhood of $\tilde{S}$. This may be made precise by the method used by Dixon (1967, § 6) for the case of special relativity, and the techniques used there may then be used to justify the above speculation for this case. A good approximation to the equations of motion is thus obtained by truncating the series (12.3) and (12.4) for $\hat{I}^{\lambda\mu}$ and $\hat{m}^{\lambda}$ after only a finite number of terms. If the highest order terms retained involve the 2^{N}-pole moments, this is known as the 2^{N}-pole moment approximation.

When this is done, the k-space integrations in (12.7) and (12.8) can be performed explicitly. Consider the contribution to the integral in (12.7) from the 2^n-pole moment of $T^{\alpha\beta}$. We first use (2.17) and (2.20) to show that

$$\int \mathrm{D}k\, k_{\nu_1} \dots k_{\nu_n} \nabla_{\kappa *} \tilde{G}_{\lambda\mu} = \nabla_{\kappa} \int \mathrm{D}k\, k_{\nu_1} \dots k_{\nu_n} \tilde{G}_{\lambda\mu}. \tag{13.1}$$

But by repeated use of (2.27) together with the Fourier inversion theorem, we also have that

$$(-\mathrm{i})^n \int \mathrm{D}k\, k_{\nu_1} \dots k_{\nu_n} \tilde{G}_{\lambda\mu} = (2\pi)^4 [\nabla_{*\nu_1 \dots \nu_n} G_{\lambda\mu}(z, X)]_{X=0}. \tag{13.2}$$

This right hand side is an ordinary tensor at z. On using (10.17) together with the interpretation of the relation (3.16) given in the paragraph following that equation, we see that in any normal coordinate system with pole at z, this tensor reduces to the repeated partial derivative

$$\partial_{\nu_1 \dots \nu_n} g_{\lambda\mu}(z).$$

The unique tensor field satisfying this is the nth *extension* of $g_{\lambda\mu}$, introduced by Veblen & Thomas (1923), and we follow them in denoting it by $g_{\lambda\mu,\, \nu_1 \dots \nu_n}$. Extensions are also discussed by Schouten (1954, ch. III, §7). Note that the first extension of a tensor is simply its covariant derivative, but that higher extensions are not simply repeated covariant derivatives. Some formulae useful in calculating such higher extensions are given in appendix 2. We have now shown that

$$(-\mathrm{i})^n \int \mathrm{D}k\, k_{\nu_1} \dots k_{\nu_n} \nabla_{\kappa *} \tilde{G}_{\lambda\mu} = (2\pi)^4 \nabla_{\kappa} g_{\lambda\mu,\, \nu_1 \dots \nu_n}. \tag{13.3}$$

The corresponding contribution from J^α requires the evaluation of $[\nabla_{*\nu_1 \dots \nu_n} \Phi_\lambda]_{X=0}$. This differs from (13.2) in that Φ_λ is not the image of any tensor field under Exp^A; instead, it is defined in terms of $F_{\alpha\beta}$ by (8.8). We deal with this by expressing it in terms of $\Phi_{\lambda\mu}$ by (12.9), and then recalling that $\Phi_{\lambda\mu} = \mathrm{Exp}^A F_{\alpha\beta}$. From (12.9), we have

$$\nabla_{*\nu_1 \dots \nu_n} \Phi_\mu - \nabla_{*\mu\nu_1 \dots \nu_{n-1}} \Phi_{\nu_n} = -\nabla_{*\nu_1 \dots \nu_{n-1}} \Phi_{\nu_n \lambda}. \tag{13.4}$$

But from (8.35) and (8.39) we get $X^\rho \Phi_\rho = 0$, which by repeated differentiation gives

$$X^\rho \nabla_{*\nu_1 \dots \nu_n \mu} \Phi_\rho + \nabla_{*\nu_1 \dots \nu_n} \Phi_\mu + n \nabla_{*\mu(\nu_1 \dots \nu_{n-1}} \Phi_{\nu_n)} = 0. \tag{13.5}$$

If we evaluate (13.5) at $X^\rho = 0$, we can use it to symmetrize (13.4) over $\nu_1 \dots \nu_n$ to give

$$[\nabla_{*\nu_1 \dots \nu_n} \Phi_\lambda]_{X=0} = \frac{n}{n+1} F_{\lambda(\nu_1,\, \nu_2 \dots \nu_n)}. \tag{13.6}$$

The corresponding integrations in (12.8) are straightforward. We thus see that the equations of motion in the 2^N-pole moment approximation are

$$\delta p_\kappa / \mathrm{d}s = \tfrac{1}{2} v^\lambda S^{\mu\nu} R_{\kappa\lambda\mu\nu} - q v^\lambda F_{\kappa\lambda} + \frac{1}{2} \sum_{n=2}^{N} \frac{1}{n!} I^{\nu_1 \dots \nu_n \lambda\mu} \nabla_\kappa g_{\lambda\mu,\, \nu_1 \dots \nu_n} + \sum_{n=1}^{N} \frac{n}{(n+1)!} m^{\nu_1 \dots \nu_n \lambda} \nabla_\kappa F_{\lambda(\nu_1,\, \nu_2 \dots \nu_n)} \tag{13.7}$$

and $$\delta S^{\kappa\lambda} / \mathrm{d}s = 2 p^{[\kappa} v^{\lambda]} + \sum_{n=1}^{N-1} \frac{1}{n!} g^{\sigma[\kappa} I^{\lambda]\rho_1 \dots \rho_n \mu\nu} g_{\{\sigma\nu,\, \mu\}\, \rho_1 \dots \rho_n} + 2 \sum_{n=0}^{N-1} \frac{1}{n!} g^{\sigma[\kappa} m^{\lambda]\rho_1 \dots \rho_n \mu} F_{\sigma\mu,\, \rho_1 \dots \rho_n}. \tag{13.8}$$

This is the final form in which we give the equations of motion. The advantages of this treatment of the multipole approximation over that of other authors were discussed in the Introduction.

In a flat space-time in a Minkowskian coordinate system, when the forces are purely electromagnetic, both the covariant derivative and extension operations reduce to partial differentiation. The above equations then reduce to those obtained by Dixon (1967) for that case. The tensorial extensions needed to evaluate (13.7) and (13.8) in the octopole approximation are given, from the results of appendix 2, by

$$\left.\begin{aligned} g_{\lambda\mu,\,\nu\rho} &= -\tfrac{2}{3}R_{\lambda(\nu\rho)\mu}, \quad g_{\lambda\mu,\,\nu\rho\sigma} = -\nabla_{(\nu}R_{|\lambda|\rho\sigma)\mu}, \\ F_{\lambda\mu,\,\nu} &= \nabla_\nu F_{\lambda\mu}, \quad F_{\lambda\mu,\,\nu\rho} = \nabla_{(\nu\rho)}F_{\lambda\mu} + \tfrac{2}{3}F^{\sigma}{}_{.[\lambda}R_{\mu](\nu\rho)\sigma}. \end{aligned}\right\} \tag{13.9}$$

We may give a general form for the gravitational terms in (13.7) and (13.8) if we work only to first order in the curvature tensor. From (A 2.4) and (A 2.7) we see that

$$g_{\lambda\mu,\,\nu_1\ldots\nu_n} = -\frac{2(n-1)}{n+1}\nabla_{(\nu_3\ldots\nu_n}R_{|\lambda|\,\nu_1\nu_2)\mu} + O(R^2) \tag{13.10}$$

if $n \geqslant 2$. For the cases $n = 2, 3$, the $O(R^2)$ terms are absent and we recover (13.9), but they are nonzero for $n \geqslant 4$.

Finally, let us see how to recover the Newtonian equations of motion from (13.7) and (13.8). These arise from the equations governing the purely spacelike components of p_κ and $S^{\kappa\lambda}$, as the equations governing p_0 and $S^{0\lambda}$ are energy equations. The recovery of Newtonian results from the latter requires a more detailed investigation than we wish to give here. We shall let latin indices run from 1 to 3, and shall consider only the purely gravitational case.

To the required order, the only component of the metric tensor which differs significantly from the Minkowskian values $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ is g_{00}. If ϕ is the Newtonian gravitational potential, we have $g_{00} = 1+2\phi$. We evaluate the moments by treating the space-time as flat and taking $n^a = 0$. Equations (5.35) and (5.36) then show that $I^{\nu_1\ldots\nu_n\lambda\mu}$ vanishes, for $n \geqslant 2$, if three or more indices are zero. Taken together with the above remarks about the metric, this shows that the only I's that contribute to the equations of motion in this approximation are $I^{a_1\ldots a_n 00}$, $n \geqslant 2$. To evaluate these, put $T^{00} = \rho$, the Newtonian mass density, and define its Newtonian moments as in (1.9). Then we obtain from § 9 that

$$I^{a_1\ldots a_n 00} = m_{a_1\ldots a_n} \tag{13.11}$$

if $n \geqslant 2$, and from (8.12) and (8.13) that

$$p_0 = m, \quad S^{a0} = m_a. \tag{13.12}$$

We also have, to this order, that

$$\Gamma^0_{a0} = \partial_a\phi, \quad \Gamma^a_{00} = \partial_a\phi, \quad R_{a0b0} = \partial_{ab}\phi. \tag{13.13}$$

The Newtonian velocity v_a, and the momentum p_a and spin S_{ab} of (1.4), can be identified with the purely spacelike components of v^α, p^α and $S^{\alpha\beta}$ respectively. However, because of the signature of our space-time metric, this identification must be performed with all the relativistic indices being superscripts, otherwise discrepancies of sign will occur. If the above values are now substituted into (13.7) for $\kappa = a$, and (13.8) for $\kappa = a$ and $\lambda = b$, we obtain the Newtonian equations (1.7) and (1.8) as required.

14. Discussion

It is unnecessary to conclude with a summary, since one is contained in outline in the Introduction, and in more detail in §§ 10 and 13 together, so that instead, it is sufficient to make some general remarks. This paper concludes the main programme of work initiated in I (Dixon 1970*a*) and continued in II (Dixon 1970*b*), giving a detailed analysis of the moment structure of the energy-momentum tensor $T^{\alpha\beta}$ and charge-current vector J^{α} of an extended body in general relativity. We saw in the Introduction the difficulties that arise if unsuitable moments are used in the equations of motion, and we there noted guidelines that suitable moments should satisfy. We have now found such moments, the reduced moments of $T^{\alpha\beta}$ and J^{α}, and have expressed the equations of motion in terms of them. In the course of doing so, it has become clear that the original guidelines we laid down are so restrictive that there is little arbitrariness left in the choice of moments. It should thus no longer be surprising that the *a priori* guesses used in the earlier theories discussed in the Introduction turned out to be unsuitable. By turning the problem around and first studying the moment structure of the body, we have been able to avoid all the difficulties of these earlier theories, and to give an explicit form for the equations of motion correct to any desired multipole order.

One slight inconsistency in notation between the three papers needs to be mentioned. In parametrizing the world line l as $z^{\lambda}(s)$, we took s to be the proper time in I and II, but in the present paper we have left it arbitrary. This choice gave rise in I to the factor χ appearing in the equations (I, 7.4) and (I, 7.5), which was introduced to simplify the treatment given there of energy transfer between the field and the body. It also caused a discrepancy of a factor of χ between the moments of J^{α} as defined in II, and as used in I. By leaving s arbitrary in the present treatment, we are leaving an undetermined scale factor in all the moments; a change of variable $s \to s'$ would multiply all the moments except p^{α}, $S^{\alpha\beta}$ and q by $\mathrm{d}s/\mathrm{d}s'$. If we take s as the proper time, we get agreement with II. If we determine s so that $v^{\lambda}u_{\lambda} = 1$, where u_{λ} is a unit vector parallel to p_{λ}, we get agreement with I. The equations of motion are invariant in form under such a change of parameter. Incorporating this factor of χ into the choice of s thus gives the advantages of the scaling of the moments used in I without the disadvantage of the scaling factor appearing explicitly. The development given in II holds also, almost without alteration, for a general choice of s.

Further consequences of the equations of motion will be developed in a later paper. This will include a treatment of energy absorption along the lines indicated in outline in I.

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Appendix 1. Summary of notation and conventions

Space-time is considered as a four-dimensional pseudo-Riemannian manifold of class C^∞, with metric tensor $g_{\alpha\beta}$ of signature -2 and Riemannian connexion $\Gamma^\gamma_{\alpha\beta}$. Covariant differentiation is denoted by ∇_α, partial differentiation by ∂_α. In repeated differentiation, the kernel ∇ or ∂ is only written once, e.g. $\nabla_{\alpha\beta}A_\gamma := \nabla_\alpha\nabla_\beta A_\gamma$. Here, as elsewhere, a colon placed before an equals sign indicates that the equation is to be regarded as defining the quantity on the left hand side. The scalar product $g_{\alpha\beta}a^\alpha b^\beta$ of two vectors is written as $a.b$. The unit tensor is denoted by A^α_β.

Symmetrization and antisymmetrization of indices is denoted by ( ) and [ ] respectively, indices to be omitted from these operations being enclosed between vertical lines, e.g.

$$A_{[\alpha|\beta|\gamma]} = \tfrac{1}{2}(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}). \tag{A 1.1}$$

Both of these symbols may be used around any number of indices. In addition, we define two special index permutations, for three and four indices respectively, by writing

$$A_{\{\alpha\beta\gamma\}} = A_{\alpha\beta\gamma} - A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} \tag{A 1.2}$$

and

$$A_{[\alpha[\beta\gamma]\delta]} = \tfrac{1}{4}(A_{\alpha\beta\gamma\delta} - A_{\gamma\beta\alpha\delta} - A_{\alpha\delta\gamma\beta} + A_{\gamma\delta\alpha\beta}). \tag{A 1.3}$$

In these, as in most of our conventions, we follow Schouten (1954).

The sign of the curvature tensor is such that the Ricci identity for a covariant vector A_α is

$$\nabla_{[\alpha\beta]}A_\gamma = -\tfrac{1}{2}R_{\alpha\beta\gamma}{}^{\delta}A_\delta. \tag{A 1.4}$$

The electromagnetic field tensor $F_{\alpha\beta}$ is taken such that in flat spacetime, with Minkowskian coordinates with metric tensor $g_{\alpha\beta} = \operatorname{diag}(1, -1, -1, -1)$, the electric and magnetic field vectors in the 3-spaces x^0 = constant are given by

$$\boldsymbol{E} = (F^{01}, F^{02}, F^{03}), \quad \boldsymbol{H} = (F^{23}, F^{31}, F^{12}). \tag{A 1.5}$$

Here, as elsewhere, 3-vectors are denoted by bold face type. The charge-current vector J^α is such that $J^\alpha = \rho v^\alpha$ for a charge distribution with density ρ and velocity v^α.

For the theory of bitensors, we follow closely the notation of DeWitt & Brehme (1960). As these quantities have tensor character at more than one point, it is necessary to indicate the point associated with each index. Unless otherwise stated, we shall always label the points involved as x and z, and use $\alpha, \beta, \ldots$ as indices at x and $\kappa, \lambda, \ldots$ at z. With this convention we may unambiguously write A^α and A^λ to denote the value of a vector field A^α at x and z respectively. If $x(u)$ is the parametric form of a geodesic joining $z = x(u_1)$ and $y = x(u_2)$, with u an affine parameter along it, the *world function biscalar* $\sigma(x, z)$ is defined by

$$\sigma(z, y) := \tfrac{1}{2}(u_2 - u_1)\int_{u_1}^{u_2} g_{\alpha\beta}(x(u))\frac{\mathrm{d}x^\alpha}{\mathrm{d}u}\frac{\mathrm{d}x^\beta}{\mathrm{d}u}\,\mathrm{d}u. \tag{A 1.6}$$

It is independent of the particular affine parameter chosen. Covariant derivatives of σ will be denoted simply by appropriate suffixes, e.g. $\sigma_{\alpha\beta\kappa} := \nabla_\kappa\nabla_{\beta\alpha}\sigma$, where, in accordance with the convention on indices, $\nabla_{\beta\alpha}$ acts at x and ∇_κ at z. The coincidence limit $x \to z$ of a bitensor field gives an ordinary tensor field, denoted by enclosing the bitensor in diamond brackets $\langle\,\rangle$.

If f and g are any two functions, the composition function $x \mapsto f(g(x))$ will be denoted by $f \circ g$, whenever this composition is meaningful. Equation numbers preceded by I or II refer to equations in papers I and II of this series respectively.

Appendix 2. Extensions and normal tensors

If a tensor field T of type (r, s) is given, then for each integer $n \geqslant 1$ there exists a unique tensor field of type $(r, s+n)$ which reduces, at the pole of any normal coordinate system, to the n-fold repeated partial derivative of T. This is known as the nth *extension* of T. The additional n tensor indices will either be suffixed to T and separated by a comma from the original indices, or written as an operator $\mathrm{E}_{\alpha_1 \ldots \alpha_n}$, thus $\mathrm{E}_{\alpha\beta} t_{\gamma\delta} = t_{\gamma\delta,\,\alpha\beta}$. The concept is due to Veblen & Thomas (1923), and much information on its properties can be found therein.

The first extension is simply the ordinary covariant derivative. To relate higher extensions to repeated covariant derivatives, we use the related concept of *normal tensors*. The nth normal tensor field $N_{\dot{\alpha}_1 \ldots \dot{\alpha}_n \dot{\beta} \dot{\gamma}}{}^{\delta}$ is the unique tensor field which reduces, at the pole of any normal coordinate system, to $\partial_{\alpha_1 \ldots \alpha_n} \Gamma^{\delta}_{\beta\gamma}$. It thus satisfies

$$N_{\dot{\alpha}_1 \ldots \dot{\alpha}_n \dot{\beta} \dot{\gamma}}{}^{\delta} = N_{(\dot{\alpha}_1 \ldots \dot{\alpha}_n)(\dot{\beta}\dot{\gamma})}{}^{\delta}, \quad N_{(\dot{\alpha}_1 \ldots \dot{\alpha}_n \dot{\beta} \dot{\gamma})}{}^{\delta} = 0, \tag{A 2.1}$$

and the first two are given, in their totally covariant form, by

$$\left.\begin{aligned} N_{\alpha\beta\gamma\delta} &= \tfrac{2}{3} R_{\alpha(\beta\gamma)\delta} \\ \text{and} \qquad N_{\alpha\beta\gamma\delta\epsilon} &= \tfrac{5}{6} \nabla_{(\alpha} R_{\beta)(\gamma\delta)\epsilon} - \tfrac{1}{6} \nabla_{(\gamma} R_{\delta)(\alpha\beta)\epsilon}. \end{aligned}\right\} \tag{A 2.2}$$

The extensions of a covariant vector v_β may be sequentially evaluated in terms of covariant derivatives and normal tensors using the recurrence relation

$$\mathrm{E}_{\alpha_1 \ldots \alpha_n} v_\beta = \nabla_{(\alpha_1 \ldots \alpha_n)} v_\beta + \sum_{r=2}^{n} \sum_{s=r}^{n} \binom{s-1}{r-1} N_{(\dot{\alpha}_1 \ldots \dot{\alpha}_r | \dot{\beta} |}{}^{\gamma} \, \mathrm{E}_{\alpha_{r+1} \ldots \alpha_s} \nabla_{\alpha_{s+1} \ldots \alpha_n)} v_\gamma, \tag{A 2.3}$$

where the $\binom{n}{r}$ are binomial coefficients. For a covariant tensor of higher degree, one set of terms such as is given here occurs for each tensor index in the obvious manner. On applying this result to the metric tensor, we get

$$g_{\beta\gamma,\,\alpha_1 \ldots \alpha_n} = 2N_{(\alpha_1 \ldots \alpha_n)(\beta\gamma)} + 2 \sum_{r=2}^{n-2} \binom{n-1}{r-1} A^{\epsilon}_{(\beta} g_{\gamma)\delta,\,(\alpha_1 \ldots \alpha_r} N_{\dot{\alpha}_{r+1} \ldots \dot{\alpha}_n) \dot{\epsilon}}{}^{\dot{\delta}}. \tag{A 2.4}$$

Note that in (A 2.3) and (A 2.4), the normal tensors only occur in the form of their partial symmetrizations $N_{(\alpha_1 \ldots \alpha_n \beta)\gamma\delta}$. These may easily be evaluated to first order in the curvature tensor as follows. By differentiating the defining relation for the curvature tensor, we get

$$\nabla_{\alpha_1 \ldots \alpha_n} R_{\dot{\beta}\dot{\gamma}\dot{\delta}}{}^{\epsilon} = 2\partial_{\alpha_1 \ldots \alpha_n [\beta} \Gamma^{\epsilon}_{\gamma]\delta} + O(\Gamma^2), \tag{A 2.5}$$

where $O(\Gamma^2)$ denotes terms at least quadratic in $\Gamma^{\gamma}_{\alpha\beta}$ and its derivatives. By considering this at the pole of a normal coordinate system, we see that it implies

$$\nabla_{\alpha_1 \ldots \alpha_n} R_{\beta\gamma\delta\epsilon} = 2N_{\alpha_1 \ldots \alpha_n [\beta\gamma]\delta\epsilon} + O(R^2), \tag{A 2.6}$$

where $O(R^2)$ denotes terms at least quadratic in the curvature tensor. It then follows from (A 2.1) and (A 2.6) that for $n \geqslant 2$,

$$N_{(\alpha_1 \ldots \alpha_n)\beta\gamma} = -\frac{n-1}{n+1} \nabla_{(\alpha_3 \ldots \alpha_n} R_{|\beta| \alpha_1 \alpha_2)\gamma} + O(R^2). \tag{A 2.7}$$

The higher order terms may be obtained if desired by including the omitted terms in the above. They are easily seen to be absent in (A 2.6) if $n = 0$ or 1, and hence also in (A 2.7) if $n = 2$ or 3.