AIP | Journal of Mathematical Physics

Motion of a body in general relativity

Robert Geroch and Pong Soo Jang

Citation: J. Math. Phys. **16**, 65 (1975); doi: 10.1063/1.522416
View online: http://dx.doi.org/10.1063/1.522416
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v16/i1
Published by the American Institute of Physics.

Motion of a body in general relativity*

Robert Geroch and Pong Soo Jang

Department of Physics, University of Chicago, Chicago, Illinois
(Received 26 August 1974)

A simple theorem, whose physical interpretation is that an isolated, gravitating body in general relativity moves approximately along a geodesic, is obtained.

1. INTRODUCTION

It is a consequence of Einstein's equation in general relativity that the divergence of the stress—energy tensor of matter vanishes, i.e., that, in physical terms, "locally, energy and momentum are conserved." One might expect, therefore, that it should be true in some sense that the motion of a body in the theory must be along a geodesic. One would like to prove some theorem in general relativity to this effect. The difficult part of obtaining such a theorem is apparently the formulation of its statement. A physical body is described in the theory by a four-dimensional region of space—time: What, then, is to be meant by "move along a geodesic?" Even the passage to an infinitesimal body does not immediately resolve the difficulty, for in this case, although one indeed obtains a unique world line for the body, the metric would be expected to become singular there: What, then, is to be meant by "this world line is a geodesic?"

A number of results suggestive of geodesic motion are known.

It is easily shown that, if the matter consists only of dust, then the world line of each dust particle must be a geodesic. This result suggests the following conjecture: The world tube of any body contains a timelike geodesic. Indeed, this conjecture is known[1] to be true for the case of a perfect fluid with isotropic pressure. Unfortunately, the conjecture is apparently false for more general sources.[2]

In an alternative approach to the problem of motion, due to Newman and his co-workers,[3] the motion of the body is described in terms of the asymptotic behavior of its gravitational field. The final equations governing this asymptotic field are indeed suggestive of geodesic motion. It appears, however, to be difficult to interpret these equations directly in terms of the appearance of the body to observers in its local neighborhood. Furthermore, the method is not immediately applicable to the case of one body moving under the influence of another, since the asymptotic analysis would require that both bodies in this case be regarded as a single system. It has been suggested[4] that both of these difficulties can be avoided, at least for the case of a black hole, by reinterpreting the equations as representing the behavior of the gravitational field near the hole.

A third approach[5] involves the passage to the limit of an infinitesimal body, i.e., the replacement of the physical body by a "line singularity" in an otherwise smooth space—time. One wishes to show, by analyzing the structure of such a singular world line, that it represents, in some sense, a geodesic. Since the metric itself is singular on this world line, one is forced to introduce some sort of averaging procedure. Apparently, the procedures available at present may not be independent of the choice of coordinates. Furthermore, recent work[6] suggests that there may even be ambiguities already in the attachment, to a smooth space—time, of the "world line of singular points."

Finally, we mention an approach due to Dixon,[7] in which one introduces a certain world-line within a gravitating body, a line which suitably generalizes the Newtonian center of mass. The acceleration of this world line is expressed as a sum of integrals over the body, where these integrals represent the interaction of the mass multipoles of the body with the curvature of space—time. Geodesic motion arises as follows: One would expect that, for the case of a "small body, with little multipole structure," these integrals will also be small, whence the center-of-mass line will be nearly a geodesic. Of course, this formulation gives, not only this geodesic limit, but also the motion of a body in detail. Consider, for example, an isolated body which is spherical and homogeneous, except for a small region of higher density, slightly displaced from the center. One expects (e.g., from the Newtonian limit) that the center-of-mass world line of such a body will not be a geodesic; the present formulation would express this acceleration in terms of integrals over the body. Yet external observers would see the body as a whole moving approximately on a geodesic. What one might like to do for this example, and what is apparently difficult to do in detail, is introduce an "average acceleration" of the entire body, rather than an "acceleration of its average position."

The purpose of this paper is to introduce still another approach to the problem of the motion of a body in general relativity. Our approach differs from those discussed above in one, apparently minor, respect: We first introduce a world line, and only then the gravitating body, rather than the other way around. One is thus able to obtain a theorem which suggests geodesic motion, which is general, and yet which is extremely simple, both to state and to prove. The disadvantages of our approach are, first, that the physical interpretation of the theorem is somewhat less direct, and, second, that the method itself is not well-suited to obtaining any further details about the motion of the body.

2. MOTION OF BODIES

We first recall some facts about the motion of a body in special relativity.

We represent our body by a nonzero, symmetric tensor field T^{ab}, its stress—energy, on Minkowski space

M, where this T is conserved: $\nabla_b T^{ab}=0$. Denote by P_a and J_{ab} $(=J_{[ab]})$ those tensor fields[8] on M with the following property: for any Killing field ξ^a on M,

$$-P_a\xi^a + J_{ab}\nabla^a\xi^b = \int_S T^{ab}\xi_b\, dS_a, \tag{1}$$

where the integral on the right extends over any spacelike 3-surface S cutting the world tube of the body, i.e., cutting the support of T. By conservation of T and Killing's equation, this integral is independent of the choice of S. Physically, P_a and J_{ab}, evaluated at a point of M, represent the momentum and angular momentum, respectively, of the body about this point as origin. From the fact that the left side of (1) must be independent of position, it follows that

$$\nabla_a P_b = 0,$$

$$\nabla_a J_{bc} = g_{a[b}P_{c]}. \tag{2}$$

This, of course, is the dependence one would expect of the momentum and angular momentum on the choice of origin.

Now suppose that our T^{ab} satisfies the following (strong) energy condition: For t_a and t'_a any future-directed timelike vectors at a point at which T^{ab} is nonzero, $T^{ab}t_at'_b$ is positive. It follows in this case from (1) (choosing for ξ^a a time-translation) that P_a is also timelike and future-directed. Define the center-of-mass world line γ of the body as the set of points of M at which $P^aJ_{ab}=0$. It is easily checked from (2) (which can be integrated explicitly) that this γ is a timelike geodesic, with tangent vector P^a.

There remains only to show that, in some sense, this center-of-mass world line γ remains "near the world tube of the body." Define the (spatial) *convex hull* of T to be the union of all segments of spacelike geodesics having both endpoints in the world tube. Consider now (1), evaluating the left side at a point p of γ, using for the S on the right the spacelike 3-plane through p orthogonal to P^a, and using for ξ^a a boost about P^a at p. For these choices, the left side of (1) vanishes. But the integral on the right is a positively weighted average, over the support of T, of position relative to p. Hence, p must lie within the convex hull of T. We conclude that the geodesic γ lies entirely within the convex hull of T. In this sense, then, a body in special relatively "moves on a geodesic."

Of course, the above result is not available in the presence of curvature, for one does not normally have enough Killing fields in that case. Our result is the following:

Theorem: Let M, g_{ab} be a space—time. Let Γ be a curve on M satisfying the following condition: For any neighborhood U of Γ, there exists a nonzero, symmetric, conserved tensor field T^{ab} on M which satisfies the energy condition, and whose support is in U. Then Γ is a timelike geodesic.

The proof consists of noting that "the nearer one is to Γ, the more nearly is the result of special relativity applicable." Fix,[9] once and for all, a flat metric $\tilde{g}_{ab}$ in some neighborhood of Γ, such that the metrics g_{ab} and $\tilde{g}_{ab}$, as well as their derivative operators ∇_a and $\tilde{\nabla}_a$, coincide on Γ. Consider a symmetric T^{ab} having support in this neighborhood. For each spacelike 3-plane (with respect to $\tilde{g}$) S, define $P_a(S)$ and $J_{ab}(S)$ by (1), where the Killing fields therein refer to $\tilde{g}$, and where the integral on the right is to be carried out over S. For each S, this $P_a(S)$ and $J_{ab}(S)$ satisfy (2), and so we obtain as before a geodesic (with respect to $\tilde{g}$), $\gamma(S)$ at a point of the convex hull (with respect to $\tilde{g}$) of T.

Now suppose that T^{ab} is conserved with respect to g. Then T^{ab} will not in general be conserved with respect to $\tilde{g}$. However, since the derivative operators coincide on Γ, $\tilde{\nabla}_b T^{ab} = (\tilde{\nabla}_b - \nabla_b)T^{ab}$ can be made as small as we wish (relative to the size of T^{ab}) by choosing the support of T^{ab} to be sufficiently small. Since the difference between the right sides of (1) for two surfaces, S and S', is given by $\int_V (\tilde{\nabla}_b T^{ab})\xi_a\, dV$ where the integral extends over the region V between S and S', this right side can also be made as small as we please. That is, the geodesics $\gamma(S)$, as S ranges over 3-planes, can all be made to be as close to each other as we wish. From this and the fact that the intersection of each S with the convex hull of the world tube contains a point of some $\gamma(S)$, we conclude that the curve Γ is as close as we wish to some geodesic (with respect to $\tilde{g}$). But this is possible only if Γ is itself a geodesic with respect to $\tilde{g}$. Since $\nabla_a = \tilde{\nabla}_a$ on Γ, Γ must therefore be a geodesic also with respect to g.

Of course, the physical interpretation of the theorem is that, for any body, "insofar as that body is sufficiently small compared with the curvature that it may be regarded as a realization of the limit implicit in the theorem, then to that extent so may it be regarded as following some geodesic Γ."

Finally, we remark that the theorem does not conflict with the standard (nongeodesic) equations for the motion of a spinning body, or of a body with a quadrupole moment. For a body satisfying the energy condition, and with spatial extension of the order of δ, its angular momentum per unit mass and quadrupole moment per unit mass cannot exceed the order of δ and δ^2, respectively. Thus, for such a body, the effects of angular momentum and quadrupole moment on its motion can be made to be as small as one wishes by choosing the body itself to be sufficiently small. The theorem, however, asserts only that Γ is a geodesic if "arbitrarily small bodies follow Γ."

*Supported in part by the National Science Foundation, Contract No. GP-34721X1, and by the Sloan Foundation.

[1]A. Taub, Proc. Natl. Acad. Sci. (USA) **48**, 1570 (1962).

[2]In Fact, it is easy to construct counterexamples even to the Newtonian version of this conjecture.

[3]See, for example, E.T. Newman, R. Posadas, Phys. Rev. **187**, 1784 (1969); J. Math. Phys. **12**, 2319 (1971); R.W. Lind, J. Messmer, E.T. Newman, J. Math. Phys. **13**, 1884 (1972).

[4]E.T. Newman, private communication.

[5]See, for example, J. Goldberg, article in *Gravitation, an Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), and references therein.

[6]See, for example, R. Geroch, J. Math. Phys. **9**, 450 (1968); B. Schmidt, J. Rel. Grav. **1**, 269 (1971).

[7]G. Dixon, Nuovo Cimento **34**, 317 (1964).

[8]Such fields exist. First note that the right-hand side of (1) defines a linear mapping from the ten-dimensional vector space of Killing fields on M to the reals. Since a Killing field is completely determined by its value, together with the value of its derivative, at any one point, there exists, for each point p, a P_a and J_{ab} at p such that (1) holds with the left-hand side evaluated at p.

[9]E. Fermi, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **31**, 21, 51 (1922).