Physics 501-20

Accelerated Detector

In special relativity, a detector with a constant acceleration follows a path given by

$$t = \frac{1}{a}\sinh(a\tau) \tag{1}$$

$$x = \frac{1}{a}\cosh(a\tau) \tag{2}$$

Here a is the acceleration and τ is the proper time along the trajectory of the detector. (we have $dt = cosh(a\tau)d\tau$; $dx = sin(a\tau)d\tau$ for small $d\tau$. Then $dt^2 - dx^2 = (cosh^2(a\tau) - sinh(a\tau)^2)d\tau^2 = d\tau^2$, which is just the expression for the proper time along the path.

Note that for τ near 0,

$$t \approx \tau \tag{3}$$

$$x \approx \frac{1}{a} + \frac{1}{2}a\tau^2 = \frac{1}{a} + \frac{1}{2}at^2 \tag{4}$$

which is just the equation for an accelerated object.

(Note that I am using units in which $c = 1$ just as I used units in the quantum parts so that $\hbar = 1$.)

The first thing is that for the detector, it is the proper time, not the time t which determines its internal dynamics.Thus for the two level system, the equations will be

$$\sigma_- = \sigma_{0-}e^{-iE\tau} = \sigma_{0-}e^{-i(E/a)\mathrm{arcsinh}(at)} \tag{5}$$

Secondly, the trajectory of the detector is $x_0 = \sqrt{\frac{1}{a^2} - t^2}$. Thus if we have such an accelerated detector, the interaction Hamiltonian will be (given again that $|\psi, 0\rangle = |\phi\rangle |\downarrow\rangle$)

$$\int H_I |\phi\rangle |\downarrow\rangle = \epsilon \left[\int e^{iE\tau(t')} \sum_i \left(A_i \partial'_t \phi_i(t', x(t')) + A_i^\dagger \partial'_t \phi_i^*(t', x(t')\right) dt' \right] \tag{6}$$

In the case where the detector was at rest, the integral over t' picked out the A_i terms because we chose ϕ_i to have only temporal components what went as $e^{-i\omega t}$. Now however, it is not just the temporal parts of $\psi(t, x)$ which are important but also the spatial terms since the position of the detector is not constant, but is a function of time.

Because the detector has proper time dependence, we can switch our integration to make the integration variable be τ rather than t.

We have $dt' = \frac{dt'}{d\tau}d\tau$ Also

$$\partial_{t'}\phi(t, x(t')) = \frac{d\tau}{dt'}\partial_\tau \phi(t(\tau), x(\tau)) = \frac{1}{\frac{dt'}{d\tau}}\partial_\tau \phi(t(\tau), x(\tau)) \tag{7}$$

and thus

$$\partial_{t'}\phi(t, x(t'))dt' = \partial_\tau\phi(t(\tau), x(\tau))d\tau \tag{8}$$

It would be really easy if we could choose our modes $\phi_i(t(\tau), x(\tau))$ such that they went like $e^{i\omega\tau}$, but were still made up solely of temporal Fourier modes with temporal dependence $e^{i\omega t}$,since then the integral over τ would be easy. Fortunately such modes exist.

I will here restrict myself to 1+1 dimensions, and to a massless ($m = 0$) field. While the calculations are far easier there, they can be carried out almost as easily for a massive field theory and in higher than 1 spatial dimensions.

What we would like is to have the modes ϕ_i go as $e^{-i\nu\tau}$. Let us look at the solutions of the field equations.

$$\partial_t^2\phi - \partial_x^2\phi = 0. \tag{9}$$

we can solve this with Fourier modes $e^{-i(\omega t-kx)}$ with $\omega = |k|$ One thus has the solutions $e^{-i|k|(t-x)}$ or $e^{i|k|(t+x)}$. Writing t and x in terms of τ we have these solutions as

$$e^{-i|k|t(\tau)-x(\tau)} = e^{-i(|k|(-e^{-a\tau})/a} \tag{10}$$
$$e^{-i|k|(t(\tau)+x(\tau)} = e^{-i(|k|/a(e^{a\tau})/a} \tag{11}$$

Which is a bit of mess. We can certainly do the required integral, but there is an easier way.

Let us define a new coordinate system, $\tau,\ \rho$ where

$$t = \frac{1}{a}\sinh(a\tau)e^{a\rho} \tag{12}$$
$$x = \frac{1}{a}\cosh(a\tau)e^{a\rho} \tag{13}$$

where $\rho = 0$ is the path of the detector. One problem is that for all ρ, $e^i a\rho$ is positive, so this new set of coordinates cover just the positive values of x. To also cover the negative values of x, define another coordinate ρ' so that

$$t = \frac{1}{a}\sinh(a\tau)e^{a\rho'} \tag{14}$$
$$x = -\frac{1}{a}\cosh(a\tau)e^{a\rho'} \tag{15}$$

These coordinates are a version of what are called Rindler coordinates, after Wolfgang Rindler, a physicist who died a year ago, and was responsible for many of the "paradoxes" that you have studied in your special relativity course. It was named that by Steve Fulling, who, after having studied Parker's cosmological quantisation papers, got interested in flat spacetime. Einstein and Rosen, in a footnote of their famous wormhole paper, had showed that a coordinate change like the above was possible in flat spacetime. Rindler had rediscovered it, and emphasized the similarity of this set of flat spacetime coordinates to the Schwartzschild coordinates used by Schwartzschild in his solution to Einstein's gravitational theory, what we now call a black hole solution. The $\rho = -\infty, \tau = \pm\infty$ surface, $(t \pm x = 0)$, is similar to the horizon of a black hole, $r = 2GM/c^2, t = \pm infty$, surfaces. $(e^{a\rho} \to r - 2GM)$.

The equation of motion of the quantum field obtained by substituting τ, ρ for t, x turns out to be

$$0 = \partial_t^2 \phi - \partial_x^2 \phi = \partial_\tau^2 \phi - \partial_\rho^2 \phi = 0 \tag{16}$$

(If the field is a massive field or is in more than 1+1 dimensions, additional functions of ρ enter the equations.) One gets the same for τ, ρ'. (if one used the massive field, the equivalence would be

$$0 = \partial_t^2 \phi - \partial_x^2 \phi + m^2 \phi = \partial_\tau^2 \phi - \partial_\rho^2 \phi + \frac{e^{2a\rho}}{a^2} m^2 \tag{17}$$

The Lagrangian action becomes

$$\frac{1}{2} \int \int [(\partial_t \phi)^2 - (\partial_x \phi)^2 + m^2 \phi^2] dx dt \tag{18}$$

$$= \frac{1}{2} \int ((\partial_\tau \phi)^2 - (\partial_\rho \phi)^2 + e^{2a\rho} m^2 \phi^2] d\rho d\tau \tag{19}$$

(and we will use $m = 0$ to simplify things) Since these equations are τ and ρ independent, we can solve them by the same harmonic trick

$$\phi_\kappa(\tau, \rho) = \frac{e^{-i|\kappa|(\tau \pm \rho)}}{\sqrt{2|\kappa|(2\pi)}} \tag{20}$$

At $\rho = 0$ this has τ dependence of $e^{-i|\kappa|\tau}$ which is exactly the time dependence we want to make the integral in the detector response trivial.

Define new coordinates,

$$U = t - x; \quad V = t + x \tag{21}$$

$$u = \tau - \rho, \quad v = \tau + \rho \tag{22}$$

$$u' = \tau + \rho'; \quad v' = \tau - \rho' \tag{23}$$

Then we have

$$U = -\frac{1}{a} e^{-au} \theta(-U) + \frac{1}{a} e^{au'} \theta(U) \tag{24}$$

$$V = \frac{1}{a} e^{av} \theta(V) - \frac{1}{a} e^{-av'} \theta(-V) \tag{25}$$

The field equation in terms of these null coordinates are

$$\partial_t^2 \phi - \partial_x^2 \phi = 2\partial_U \partial_V \phi = 2e^{-a(v-u)} \partial_u \partial_v \phi = e^{2a\rho} (\partial_\tau^2 \phi - \partial_\rho^2 \phi) = 0 \tag{26}$$

Thus the conjugate momentum along the $\tau = t = 0$ is

$$\pi = \partial_t \phi = e^{-a\rho} \partial_\tau \phi \tag{27}$$

and the inner product is

$$< \phi, \phi' >= \int e^{-a\rho} (\phi^* \partial_\tau \phi' - \partial_\tau \phi^* \phi') dx \tag{28}$$

$$= \int (\phi^* \partial_\tau \phi' - \partial_\tau \phi^* \tau \phi') d\rho \tag{29}$$

since along $\tau = 0$ the relation between dx and $d\rho$ is

$$dx = sin(a\tau) e^a \rho dt + cosh(a\tau) e^{a\rho} d\rho = e^{a\rho} d\rho \tag{30}$$

Again, the inner product is independent of the time τ so The inner product will be the same.

For ρ' (x¡0), we get the same with $\rho \to \rho'$.

Let us look at these "Rindler modes" with the v dependence.

$$e^{-i|\kappa|(\tau+\rho)} = e^{-i|\kappa|v} = (\frac{1}{a} e^{av})^{-i\frac{|\kappa|}{a}} \tag{31}$$

$$= V^{-i\frac{|\kappa|}{a}} \tag{32}$$

There is a theorem that if one makes a function out of only positive frequencies $f = \int_0^\infty alpha_\omega e^{-i\omega t} d\omega$, then f must be analytic for $Im(t) < 0$. ($e^{-i\omega t} = e^{-i\omega Re(t)} e^{-omegaIm(t)}$ which goes to 0 for large ω.

Let us look at the above modes. For the V modes, since $V = t - x$, as V increases, so does t (as long as say $x(t)$ is timelike).

The problem with this function is at $V = 0$ where the function has a singularity. We can make it analytic in the upper $Im(V)$ by deforming the integral path of V into the lower plane Ie, $\lim_{\lambda \to +0} (V + i\lambda)^{\pm i \frac{|\kappa|}{a}}$ is analytic in the upper half $Im(V)$ half plane and is made up of Fourier components $e^{-i\omega V}$. This is true independent of the sign of $\pm|\kappa|$.

Let us look at the functions

$$\phi_\Omega(V) = (V + i\lambda)^{i\Omega/a} \tag{33}$$

$$\phi_\Omega(U) = (U + i\lambda)^{i\Omega/a} \tag{34}$$

where Ω is real but of arbitrary sign. The $i\lambda$ notation indicates that λ is positive, and that one takes the limit of λ goes to 0 from the positive direction. The singularity occurs at $V = -i\lambda$ and as a result the functions are analytic and bounded in the whole positive imaginary V plane for all real Ω. The norm is

$$\phi_\Omega, \phi'_\omega >= i \int (\phi^*_\Omega \partial_t \phi'_\Omega - \phi'_\Omega \partial_t \phi^*_\Omega) dx \tag{35}$$

$$= i \int (\phi^*_\Omega \partial_V \phi'_\Omega - \phi'_\Omega \partial_V \phi^*_\Omega) dV \tag{36}$$

since $\partial_t V = \partial_x V = 1$.

$$\phi_\Omega, \phi'_\omega >= i \int (\phi^*_\Omega \partial_t \phi'_\Omega - \phi'_\Omega \partial_t \phi^*_\Omega) dx \tag{37}$$

$$= i \int (\phi^*_\Omega \partial_v \phi'_\Omega - \phi'_\Omega \partial_v \phi^*_\Omega) dv + \int (\phi^*_\Omega \partial_v \phi'_\Omega - \phi'_\Omega \partial_{\tilde{v}} \phi^*_\Omega) d\tilde{v} \tag{38}$$

where $v = \tau + \rho$ and $\tilde{v} = \tau - \rho'$. Now,

$$\phi_\Omega(V + i(\lambda = +0)) = V^{i\Omega/a}\theta(V) + e^{-\pi\Omega/a}(-V)^{i\Omega/a} \tag{39}$$

since near 0 the phase of V goes from 0 radians to π radians as V goes from positive to negative values of V. Thus the phase of V, the imaginary part of $ln(V)$ goes from 0 to π as V goes from positive to negative values. Since the phase is multiplied by $i\Omega/a$, we get that the amplitude for negative values of V is smaller than positive by $e^{-\pi\Omega/a}$. If Ω is positive, then the amplitude for negative V is exponentially smaller than for positive V. If Ω is negative, then negative amplitudes are exponentially larger than for positive V.

We can now evaluate the norm.

$$\begin{aligned} < \phi_\Omega \quad &, \quad \phi'_\Omega > \\ &= \int_0^\infty (aV)^{-i\Omega/a}\partial_V (aV)^{i\Omega'/a} - \partial_V (aV)^{-i\Omega/a}(aV)^{i\Omega'/a})dV && (40) \\ &+ \int_{-\infty}^0 e^{-\pi(\Omega+\Omega')/a}(a|V|)^{-i\Omega/a}\partial_V (a|V|)^{i\Omega'/a} - \partial_V (a|V|)^{-i\Omega/a}(a|V|)^{i\Omega'/a})dV && (41) \\ &= i(1 - e^{-\pi(\Omega+\Omega')/a}a^{i(\Omega'-\Omega)}i(\Omega + \Omega') \int_0^\infty \frac{|V|^{i(\Omega-\Omega')/a}}{|V|} d|V| && (42) \\ & && (43) \end{aligned}$$

But

$$\int_0^\infty |V|^{i(\Omega-\Omega')/a} \frac{d|V|}{|V|} = \int_{-\infty}^\infty e^{i(\Omega-\Omega')\zeta/a} d\zeta = 2\pi\delta((\Omega - \Omega')/a) \tag{44}$$

where $|V| = e^\zeta$.

Thus

$$< \phi_\Omega, \phi_{\Omega'} >= 2\pi a\delta(\Omega - \Omega')(1 - e^{-\pi(2\Omega)/a} = 2\pi\delta(\Omega - \Omega')\frac{\sinh(\pi\Omega/a)}{e^{-\pi\Omega/a}} \tag{45}$$

Thus the Normalisation factor for these modes is $\frac{N=e^\pi\Omega/(2a)}{\sqrt{(2\Omega\sinh(\pi\Omega/a)}}$. (since both $\Omega$and $\sinh(\pi\Omega/a)$ are odd functions of Ω, the quantity under the square root is always positive)

For positive Ω, this mode is concentrated in the right Rindler Wedge ($V > 0$). for negative Ω it is concentrated in the left wedge.

One can go through exactly the same procedure for the U modes. One gets

$$\phi_\Omega = \frac{1}{\sqrt{2\Omega\sinh(\pi\Omega/a)}}(e^{\pi\Omega/(2a)}\theta(U) + e^{-\pi\Omega/(2a)}\theta(-U))|aU|^{-i\Omega/a} \tag{46}$$

so again for positive Ω the mode is dominant in the sector $U > 0$ and is smallest in the $U < 0$ sector. For $\Omega < 0$ the opposite is again true.

Since each of these positive norm modes can be written in terms of the Minkowski positive norm "Hamiltonian-diagonalisation" modes, the annihilation operators of these modes will be linear combinations of the Minkowski "Hamiltonian diagonalisation" Annihilation operators and have the same vacuum state $|0\rangle$. So, let us choose our ϕ_i to be these modes.

$$\Phi = \int_{-\infty}^{\infty} \frac{e^{\pi\Omega/2a}}{\sqrt{2\Omega \sinh(\pi\Omega/2a) 2\pi}} \left[A_{\Omega v}(e^{-i\Omega(\tau+\rho)} + A_{-\Omega u} e^{i\Omega(\tau-\rho)})\right] + HC; \quad x > 0 \tag{47}$$

$$= \int_{-\infty}^{\infty} \frac{e^{\pi\Omega/2a}}{\sqrt{2\Omega \sinh(\pi\Omega/2a) 2\pi}} \left[A_{-\Omega v}(e^{-i\Omega(\tau+\rho')} + A_{\Omega u} e^{i\Omega(\tau-\rho')})\right] + HC; \quad x < 0 \tag{48}$$

The state defined by

$$A_{\Omega u}|0\rangle = A_{\Omega v}|0\rangle = 0 \tag{49}$$

is exactly the same as the vacuum state defined by the usual Minkowski Annihilation operators. Note that these states are defined for all Ω, not just positive values. This is another example that "positive frequency" is NOT a sensible criterion for defining the modes corresponding to annihilation and creation operators.

This can be inserted into the expression for the first order amplitude for the detector. Since the detector lives solely in the region x¿0, we only need the expression for the detector in the right hand wedge Let us choose the state to be the Minkowski vacuum state. This is the state where particle detectors at rest seen nothing. Choosing the initial state of the field to be the Minkowski vacuum state, the annihilation operators all give 0 on that vacuum state. The detector is located at $\rho = 0$. If T is large then the integral will pick out Ω only near -E.

$$|\psi, 0\rangle = |0\rangle |\downarrow\rangle \tag{50}$$

$$|\delta\psi, T\rangle \approx -i\epsilon E \frac{e^{-\pi E/2a}}{\sqrt{2\pi E \sinh(\pi E/a)}} \int_0^T e^{iE\tau} e^{-i\Omega\tau} A^{\dagger}_{-\Omega} |0\rangle \, d\Omega d\tau \tag{51}$$

The probability of detection will be

$$P_{\uparrow} = \langle \delta\psi, T| \, |\uparrow\rangle \langle\uparrow| \, |\delta\psi, T\rangle \tag{52}$$

$$= \epsilon^2 \frac{e^{-2\pi E/a}}{(1 - e^{-2\pi E/a})} \frac{E}{4\pi} \int | \int_0^T e^{i(E-\Omega)\tau} d\tau|^2 d\Omega \tag{53}$$

The first term is expected as the probability should grow as ϵ^2. The second term is just the Einstein-Bose thermal factor with temperature of $\frac{a}{2\pi}$. The third is

$$| \int_0^T e^{i(E-\Omega)\tau} d\tau|^2 = \left(\frac{sin((E-\Omega)T/2)}{E-\Omega}\right)^2 \tag{54}$$

and

$$\int \left(\frac{sin((E-\Omega)T/2)}{E-\Omega} \right)^2 d\Omega = \pi T/2 \tag{55}$$

Ie, the probability grows linearly in time, which is what one would expect of a random excitation probability.

The detector is excited at a constant rate, and with a factor that is just the thermal factor times a "cross section" for detection in flat spacetime.