Global coordinates for Schwarzschild black holes

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A variety of historical coordinates in which the Schwarzschild metric is regular over the whole of the extended spacetime are compared and the hypersurfaces of constant coordinate are graphically presented. While the Kruscal form (one of the later forms) is probably the simplest, each of the others has some interesting features.

For years after Schwarzschild[1] found a solution for spherically symmetric metrics to Einstein equations,

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{1}$$

the status of the singularity at $r = 2M$ (in units where $c = 1$ $G = 1$) confused many, including Einstein[2]. It was only in 1933, when Lemaître[8] found his coordinate transformation that made the metric regular across the horizon, that he explicitly stated that that singularity in the metric was an artifice introduced because of the coordinates that Schwarzschild had used. It had already been recognized by Lanczos in 1922 that the status of singularities in a metric was unclear because singularities could be introduced by making a singular choice of coordinates. However, the application of this to the $r = 2M$ singularity was not appreciated.

This paper will look at a number of these coordinate transformation, in particular with the purpose of looking at a variety of coordinate systems which are regular in the whole of the extended Schwarzschild spacetime. For some (like Schild's and Kruscal's coordinates) these are universal coordinates that were actually published. In other cases they take historical coordinates which are regular across one or the other of the Schwarzschild horizons, and make turn them into universal (i.e., covering the whole of the extended Schwarzschild spacetime). Since in a large number of cases, the single horizon coordinates were discovered long before Schild's coordinates, this is an exercise in alternate reality– what could have so easily happened if only the generators of those coordinate systems had recognized what they had.

In all cases, I will also graph the constant-time hypersurfaces for the various coordinates. For the spacetime coordinates which I will choose to do the graphing in, I choose the Kruscal coordinates, since they are the best known of the universal coordinates.

In 1921, both Gullstrand and Painlevé[7] had found new, spherically symmetric solutions to Einstein's equation,

$$ds^2 = (1 - \frac{2M}{r})d\tau_\pm^2 \pm 2\sqrt{\frac{2M}{r}}d\tau_\pm dr - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{2}$$

In the following I will refer to this as the PG form of the metric. Both Painlevé and Gullstrand thought that this metric proved that Einstein's equations were inconsistent, in that two different solutions to the spherically symmetric solution thus existed encompassing different physics (different solutions as functions of the time and the radius for geodesics). They did not recognize that this solution is simply a coordinate transformations of Schwarzschild's solution, with

$$t = \tau + 2\sqrt{2Mr} + 2M\ln\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right) \tag{3}$$

nor did they recognize the implication for the Schwarzschild singularity, believing that coordinates themselves held physical significance.

[In Kruskal's[10] paper, the claim is made that Kasner[3] in 1921 also showed that the $r = 2M$ singularity was just a coordinate singularity. This is not true. Kasner embedded the Schwarzschild solution into a 6 dimensions (signature 4+2) flat spacetime but that embedding is singular at $r = 2M$– it covers only the region $r > 2M$.]

In 1922, Eddington[4] also found an explicit coordinate transformation

$$t = \tilde{t}_\pm \pm 2M\ln(\frac{r - 2M}{2M}) \tag{4}$$

which gave the metric

$$ds^2 = (1 - \frac{2M}{r})(d\tilde{t}_\pm + dr)^2 \pm 2d\tilde{t}_\pm dr - 2dr^2 - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{5}$$

which is regular (the metric tensor is regular at $r = 2M$ as is its inverse) at $r = 2M$ but he did not recognize (or at least did not comment on) the implication that this had for the so called Schwarzschild singularity. This coordinate transformation and metric were rediscovered in 1954 by Finkelstein[5] who certainly did recognize that this implied that the Schwarzschild singularity was purely a coordinate artifact. This metric is, however, not what we now call the Eddington-Finkelstein form of the metric.

What is now called the Eddington-Finkelstein (EF) form of the metric is obtained from their form by replacing $\tilde{t}_\pm$ by $u_\pm = \tilde{t}_\pm - (\pm r)$ to give

$$ds^2 = (1 - \frac{2M}{r})(du_\pm)^2 \pm 2du_\pm dr - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{6}$$

This null form seems to have first been given by Penrose in 1965 [6]), who wrongly ascribed it first to Finkelstein, and later to both Eddington and Finkelstein. This error propagated into the textbooks. I will call these the EF coordinates, for Eddington, Finkelstein since that is how they are usually referred to. The original cordinates which they actually found I will call the OEF for "Original Eddington Finkelstein".

The PG metric and the EF metric are coordinate transformations of each other, with a transformation that is regular for all values of $r > 0$. In particular, if we take

$$\tau = v - r + \sqrt{r} \tag{7}$$

we turn the PG into the EF form of the metric.

In 1933, Lemaître, concerned about cosmological solutions to Einstein's equations, introduced his form of the Schwarzschild metric. He was interested in the solution in which one embeds a Schwarzschild solution in a De-Sitter universe, but also took the limit as the cosmological constant was zero.

$$ds^2 = d\tau_\pm^2 - \frac{2M}{r(\sigma_\pm - \tau_\pm)} d\sigma_\pm^2 - r(\sigma_\pm - \tau_\pm)^2(d\theta^2 - \sin(\theta)^2 d\phi^2) \tag{8}$$

where $\frac{2}{3} r^{\frac{3}{2}} = \pm\sqrt{2M}(\sigma_\pm - \tau_\pm)$. Lemaître was the one that explicitly showed, in passing, that this was simply a coordinate transformation of the PG metric, and that the PG metric itself was just a coordinate transformation of Schwarzschild's form.

In Lemaître's form of the metric, the time $\tau_\pm$ is the proper time along a timelike geodesic which has zero velocity at infinity, and $\sigma_\pm$ is constant along that timelike geodesic, and designates the time at which the geodesic hits $r = 0$. The $\pm$ designates whether that geodesic is ingoing (r decreases with τ for the + sign)) or an outgoing geodesic (with the - sign).

As is well known, all three forms of the metric (PG,EF, and Lemaître) demonstrate that the Schwarzschild singularity is a coordinate artifact, and that the metric is regular (has a well defined inverse everywhere including at $r = 2M$) they all come in two forms which cover different regions of an extended spacetime. All cover the exterior (r¿2M) but by the change of a sign, they also cover distinctly different regions corresponding to $r < 2M$. The different coordinates, designated by $\pm$ in the various cases, cover only parts of the possible extensions of the Schwarzschild spacetime.

Is there a set of coordinates for which the only singularities occur at r=0, and in the metric is regular and complete (i.e., all geodesics end either at the equivalent of $r = 0$ or $r = \infty$ in those coordinates? The answer is of course yes. There are an infinite number of such coordinates, but I will look at some of the historically based ones. I will first look ones which were actually proposed, and then at some, based on the above coordinates which cover one of the horizons, which could have been discovered.

I will also choose units such that $2M = 1$. This factor can be reinserted by dimensional arguments.

The best known is the Szekeres-Kruscal form discovered in 1960, but the first universal coordinates were given by Synge[9] in 1950, an accomplishment which has in general sunk into the seas of forgetting.

To find Synge's transformation, write the Schwarzschild metric in terms of the proper distance to the horizon

$$R = \int_1^r \frac{dr}{\sqrt{1 - \frac{1}{r}}} = \sqrt{r-1}\sqrt{r} + \mathrm{asinh(r - 2M)} = \sqrt{\mathrm{r} - 1}\left(\sqrt{\mathrm{r}} + \frac{\mathrm{asinh}(\sqrt{\mathrm{r} - 1})}{\sqrt{\mathrm{r} - 1}}\right) \tag{9}$$

We have

$$1 - \frac{1}{r} = 2\frac{R^2}{r\left(\sqrt{r} + \frac{\mathrm{asinh}(\sqrt{\mathrm{r}-1})}{\sqrt{\mathrm{r}-1}}\right)^2} \tag{10}$$

and

$$ds^2 = F(r(R))R^2\frac{1}{4}dt^2 - dR^2 - r(R)^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{11}$$

where

$$F(r) = \frac{4}{r\left(\sqrt{r} + \frac{\mathrm{asinh}(\sqrt{\mathrm{r}-1})}{\sqrt{\mathrm{r}-1}}\right)^2} \tag{12}$$

The function $F(r)$ looks singular at $r = 1$ but is not. $\sqrt{r}$ is analytic for $r > 0$ while the function $\frac{\mathrm{asinh}(\sqrt{\mathrm{r}-1})}{\sqrt{\mathrm{r}-1}}$ is also an analytic function of r everywhere for $r > 0$. It is an even function in the argument $\sqrt{r-1}$ and is thus a function of $r - 1$ and is analytic in r for $r > 0$. $F(r)$ is also monotonic in r and thusR^2 is an analytic monotonic function of r for $r > 0$. The inverse, $r(R)$, is thus also an analytic function of R.

Also $F(r = 1) = 1$ and we can thus write the metric as

$$ds^2 = (F(r(R)) - 1)R^2\frac{1}{4}dt^2 + R^2dt^2 - dR^2 - r(R)^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{13}$$

Now defining

$$T = R\sinh(t/2) \tag{14}$$

$$\xi = R\cosh(t/2) \tag{15}$$

and thus $R^2 = \xi^2 - T^2$, we have the regular metric

$$ds^2 = (F(r(R)) - 1)(\xi dT - Td\xi)^2 + dT^2 - d\xi^2 - r(\sqrt{T^2 - \xi^2})^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{16}$$

This metric is singular for $T^2 - \xi^2 = \pi$ (which corresponds to $r = 0$) but is regular everywhere else. This is the Synge form of the Schwarzschild metric, the first of the metric forms whose coordinates cover all of the analytically extended spacetime (all geodesics either end in a genuine singularity, corresponding to one of the $r = 0$ singularities, or extend to infinity.) This metric has some interesting features. The lines of ξ, θ, ϕ constant are not necessarily timelike lines. for ξ sufficiently large and r sufficiently small, $F((r) - 1)\xi^2 + 1$ can be negative of $r < 1$ and thus the line becomes spacelike.

While Synge recognized that these coordinates covered an extended patch of the Schwarzschild spacetime, he went further and argued that it also represented an extension even to the regime $r < 0$ of Schwarzschild, apparently not recognizing that $r = 0$ was a true singularity (curvature scalars go to infinity) of the spacetime. Because of the obscurity of both the arguments and of the journal, Synge's accomplishments have largely vanished from the textbooks and literature.

The later Szekeres-Kruskal[10] metric can be formed in a similar way. Define

$$ds^2 = G(r(\rho))(\rho^2\alpha^2dt^2 - d\rho^2) - r(\rho)^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{17}$$

where α is a constant. This leads to

$$\frac{d\rho}{dr} = \alpha\frac{\rho}{1 - \frac{1}{r}} \tag{18}$$

or

$$\rho = (r - 1)^\alpha e^\alpha r \tag{19}$$

Choosing $\alpha = \frac{1}{2}$ we have

$$ds^2 = e^{-\frac{r(\rho)}{2}}\frac{1}{r}(\rho^2(\frac{dt}{4M})^2 - d\rho^2) - r(\rho)^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{20}$$

Defining

$$\tau = \rho\sinh(\frac{t}{2}) \tag{21}$$

$$\chi = \rho\cosh(\frac{t}{2}) \tag{22}$$

we get the Szekeres/Kruskal metric

$$ds^2 = e^{\frac{-r}{2}}\frac{1}{r}(d\tau^2 - d\chi^2) - r(\tau^2-\chi^2)^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{23}$$

There is another way of arriving at the same result. Writing the null EF metric

$$ds^2 = (1-\frac{1}{r})du_\pm^2 \pm 2du_\pm dr - -r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{24}$$

with

$$u_\pm = t \pm (r + \ln(r-1) \tag{25}$$

to give

$$r - 1 = e^{\frac{u_+ - u_- - 2r}{2}}\frac{1}{r} - r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{26}$$

to give

$$ds^2 = e^{-r}\frac{1}{r}(e^{\frac{u_+}{2}}du_+)(e^{-\frac{u_-}{2}}du_-) - r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{27}$$

Defining $U_\pm = \pm 4Me^{\pm u_\pm}$ and

$$\tau = (U_+ + U_-)/2 \tag{28}$$

$$\chi = (U_+ - U_-)/2 \tag{29}$$

we obtain exactly the Szekeres-Kruskal metric. Ie, in this approach, we write the metric using both the advanced and retarded forms of the coordinates. This creates a metric in which the non-angular terms of the metric are multiplied by $r-1$. Since the difference between the advanced and retarded coordinates contains a term proportional to $\ln(r-1)$, one can, by exponentiating those advanced and retarded coordinates, absorb that common factor of $r-1$ and create a regular metric at $r=1$.

This second procedure for finding the SK coordinates also allows us to carry out the same procedure for the PG metric Defining the two sets of coordinates by

$$\tau_\pm = t \pm (2\sqrt{r} + \ln\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right) \tag{30}$$

we have

$$d\tau_\pm^2 \pm \frac{\sqrt{\frac{1}{r}}}{1-\frac{1}{r}}d\tau_\pm dr - r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{31}$$

In terms of these "times" we have

$$ds^2 = -\frac{r-1}{4r}(d\tau_+^2 + d\tau_-^2) + \frac{(r)^2-1}{r}d\tau_+ d\tau_- - r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{32}$$

Defining $\Xi_\pm = e^{\tau_\pm/2}$ and $y=\sqrt{r}$, we have

$$ds^2 = (2M^2)\left[-\frac{e^{-4y}(y+1)^4}{y^2}(\Xi_+^2 d\Xi_-^2 + \Xi_-^2 d\Xi_+^2) + -2e^{-2y}\frac{(y^2+1)(y+1)^2}{y^2}d\Xi_+ d\Xi_-\right] \tag{33}$$

where $r(\Xi_+\Xi_-)$ is defined by

$$\Xi_+\Xi_- = \frac{y-1}{y+1}e^{2y} \tag{34}$$

This is again a regular metric everywhere where $r>0$ ($\Xi_+\Xi_- > -1$). It retains the feature of the PG metric that the surfaces Ξ_+ =const or Ξ_- =const are flat spacelike surfaces– i.e., it foliates the extended Schwarzschild spacetime with a series of intersecting flat spatial slices.

As a another historical example, we can look at a coordinate system related to the global embedding of the Schwarzschild metric found by Fronsdal[11]. Define the function $\hat{R}$ by

$$\hat{R}^2 = 4(1-\frac{1}{r}) \tag{35}$$

$\hat{R}$ runs from $-\infty$ ($r=0$) to 0 ($r=\infty$). Then we can write

$$ds^2 = \hat{R}^2\left(\frac{dt}{2}\right)^2 - d\hat{R}^2 - \frac{1+r+r^2+r^3}{r^3}dr^2 - r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{36}$$

$$= \hat{R}^2\left(\frac{dt}{2}\right)^2 - d\hat{R}^2 - \left(1+\left(\frac{1}{(1-(\frac{\hat{R}^2}{4}}\right)^4\right)\hat{R}^2 d\hat{R}^2 - \frac{1}{1-\frac{\hat{R}^2}{4}}(d\theta^2+\sin(\theta)^2 d\phi)^2 \tag{37}$$

As before, define

$$\Theta = \hat{R}\sinh(\frac{t}{2}) \tag{38}$$

$$Y = \hat{R}\cosh(\frac{t}{2}) \tag{39}$$

$$\hat{R}^2 = Y^2 - \Theta^2 \tag{40}$$

This gives

$$ds^2 = d\Theta^2 - dY^2 - (YdY - \Theta d\Theta)^2\left(\frac{1}{1-\frac{Y^2-\Theta^2}{4}}\right)^4 - \frac{1}{1-\frac{Y^2-\Theta^2}{4}}(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{41}$$

These are related to the global embedding of the Schwarzschild metric in a 6-dimensional flat spacetime, first suggested by Fronsdal[11]. Defining the Z coordinate by

$$Z = \int^r \frac{r'^2+r'+1}{r'^3}dr' \tag{42}$$

the metric becomes

$$ds^2 = d\Theta^2 - dY^2 - dZ^2 - dr^2 - r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{43}$$

with the above definition of Θ, Y, Z as functions of t, r giving the embedding functions of the 4 dimensional surface in the 6 dimensional flat spacetime.

In 1965 Israel[12], rediscovered in 1971 by Newman and Pajerski[13], found another global coordinate system, which has an advantage over the Kruscal form in that the coordinate transformation can be written out with the new coordinates being an explicit (not implicit) form of the Schwarzschild coordinates, unlike the Kruscal form. One of the coordinates is the same as the null coordinate of the Kruscal metric. This coordinate I will take as U. (an equivalent one exists for V)

$$U = e^{\frac{-t+r+\ln(r-1)}{2}} = e^{\frac{-t+r}{2}}\sqrt{r-1} \tag{44}$$

while the other coordinate, designated by z is

$$z = e^{\frac{t-r+2M\ln(r-1)}{2}} = \sqrt{r-1}e^{\frac{t-r}{2}} \tag{45}$$

or

$$r = (1+zU) \tag{46}$$

$$t = (\ln(\frac{z}{U}) + 1 + \frac{z}{U}) \tag{47}$$

The surface $U = const$ is a null hypersurface, the same as the SK hypersurface, while $z =$ const is a timelike hypersurfaces unless $z = 0$ in which case it is a null hypersurfaces. The metric in these coordinates is

$$ds^2 = \frac{z^2}{r}dU^2 + 2dUdz + r^2(d\theta^2+\sin(\theta)^2 d\phi^2) \tag{48}$$

where r is given by the above explicit function of U, z. I will call these the IN coordinates.

The primary advantage of those coordinates is that one can write the metric in the new coordinates explicitly in terms of these new coordinates themselves. In both of the previous examples, one has to invert a relation between the new coordinates and r which cannot be written explicitly in terms of elementary functions.

While the above are historical global coordinates, the Painlevé-Gulstrand, the Lemaître and the original Eddington Finkelstein metrics all allow us to simply define global coordinates by using the retarded and advanced coordinates. The difference between them is proportional to $\ln(r-1)$ near the horizon, and this can be used to generate global coordinates.

Let us start with the Painlevé-Gulstrand coordinates. We have

$$\frac{\tau_+ - \tau_-}{2} = -2\sqrt{r} - ln(r-1) + 2\ln(\sqrt{r}+1) \tag{49}$$

or

$$e^{-\frac{\tau_+ - \tau_-}{2}} = r - 1\left(\frac{e^{\sqrt{r}}}{(\sqrt{r}+1)^2}\right) \tag{50}$$

Defining

$$\mathcal{T}_1 = e^{\frac{-\tau_+}{2}} \tag{51}$$

$$\mathcal{T}_2 = e^{\frac{\tau_-}{2}} \tag{52}$$

we have

$$ds^2 = -(\mathcal{T}_2^2 d\mathcal{T}_1^2 + \mathcal{T}_1^2 d\mathcal{T}_2^2) + d\mathcal{T}_1 d\mathcal{T}_2 - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{53}$$

$$\mathcal{T}_1\mathcal{T}_2 = \frac{r-1}{(\sqrt{r}+1)^2} e^{-2\sqrt{r}} \tag{54}$$

The surfaces $\mathcal{T}_1$ or $\mathcal{T}_2$ constant are spacelike surfaces (except they are null along along the horizon) In keeping with the Painlevé-Gullstrand metrics these surfaces are also flat surfaces. Ie, these coordinates foliate the complete Schwarzschild metric into flat spatial hypersurfaces.

For the Lemaître metric, if we use $\tau_\pm$, we get exactly the same global coordinates as for the above Painlevé-Gullstrand metric. We can instead use the exponential of the $\sigma_\pm$ coordinates to cover the complete Schwarzschild metric.

The times $\tau_\pm$ are precisely the same as for the Painlevé-Gulstrand metric, and are the proper time along a freely falling geodesic which starts with zero velocity at infinity. $\sigma_\pm$ constant define those geodesics.

$$\Sigma_\pm = exp(\left(\frac{1}{2}\left(\frac{2}{3}r^{\frac{3}{2}} + \sqrt{r} + \ln\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right) \pm t\right)\right) \tag{55}$$

from which we find

$$\Sigma_+\Sigma_- = \frac{r-1}{(\sqrt{r}+1)^2} e^{\frac{2}{3}r^{\frac{3}{2}}+\sqrt{r}} \tag{56}$$

$$\frac{\Sigma_+}{\Sigma_-} = e^t \tag{57}$$

and the metric becomes

$$ds^2 = e^{-\frac{2}{3}r^{\frac{3}{2}}+\sqrt{r}}(\sqrt{r}+1)^2(\Sigma_+^2 d\Sigma_-^2 + \Sigma_-^2 d\Sigma_+^2 - 2d\Sigma_- d\Sigma_+) - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{58}$$

In this case the surfaces of either Σ_+ or Σ_- constant are timelike surfaces and the lines in those surfaces of θ and ϕ constant are time-like geodesics (except when $\Sigma_\pm = 0$ when they are null geodesics)

Finally, we can use the original Eddington-Finkelstein coordinates to define global coordinates by

$$\tilde{T}_1 = e^{\frac{\ln(\frac{r-2M}{2M})-t}{2}} = \sqrt{r-1}e^{-\frac{t}{2}} \tag{59}$$

$$\tilde{T}_2 = e^{\frac{\ln(r-1)+t}{2}} = \sqrt{r-1}e^{\frac{t}{2}} \tag{60}$$

or

$$r = (1 + \tilde{T}_1\tilde{T}_2) \tag{61}$$

$$t = \ln(\frac{\tilde{T}_2}{\tilde{T}_1}) \tag{62}$$

to give

$$ds^2 = -\left(\frac{r+1}{r}(\tilde{T}_2^2 d\tilde{T}_1^2 + \tilde{T}_1^2 d\tilde{T}_2^2) + \frac{r}{r+1}d\tilde{T}_2 d\tilde{T}_2\right) + (1 + \tilde{T}_1\tilde{T}_2)(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{63}$$

I will call these the OEF coordinates.

This global metric shares with the IN metric that the metric in these new coordinates can be explicitly written in terms of the new coordinates. It is probably the simplest coordinate transformation which gives a global coordinate system, although the metric does not take as neat a form as for the SK coordinates. The surface $\tilde{T}_1$ constant is spacelike everywhere except at r=2M where it is null, and similarly for $\tilde{T}_2$. Had Eddington realized what he had in his coordinates, 50 years of confusion about the Schwarzschild metric and the horizon might well have been alleviated although the double spacelike nature of these coordinates could well have confused many, just as the fact that r becomes a timelike coordinate for $r < 2M$ has led to much nonsense being written about the Schwarzschild metric. That coordinates are arbitrary labels for spacetime points and have no physical significance in and of themselves is a lesson Einstein learned early (see his response to Painlevé[2]), but is one which has continued to confuse the unwary ever since.

I. RELATIONS BETWEEN COORDINATES

Since the SK coordinates are the standard coordinate patch for covering all of Schwarzschild spacetime, let us compare the other coordinate systems to the SK coordinates graphically.

Let us first look at the generalized PG coordinates axes in the SK coordinates. The extended PG coordinate surfaces of constant $Xi_\pm$ will be plotted in SK coordinates. Using the SK coordinates $U = \tau - \rho$ and $V = \tau + \rho$ we have

$$\frac{\Xi_+}{\Xi_-} = e^{\frac{t}{2M}} = \frac{V}{U} \tag{64}$$

$$\Xi_+\Xi_- = \frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1} e^{2\sqrt{\frac{r}{2M}}} \tag{65}$$

$$UV = (\frac{r}{2M} - 1)e^{r/2M} \tag{66}$$

Ie, $\Xi_+\Xi_-$is a function of UV given parametrically by the last two equations.

The diagram indicates the graph of constant Ξ_+ and Ξ_- spacelike hypersurfaces for a few values of each.

Note that as $r \to \infty$, both Ξ_+ and Ξ_- (for suitable values) asymptote to the same line. in the UV plane. Ie, the Ξ_+, Ξ_- coordinates become degenerate as $r \to \infty$.

Let us now plot the Synge coordinates in the SK coordinates. The surfaces of constant Synge time T are given in terms of the SK coordinates parametrically by

$$\frac{V+U}{2}(T) = \frac{Te^{\frac{r}{2}}}{\sqrt{r} + \frac{\text{asinh}(\sqrt{\text{r}-1})}{\sqrt{\text{r}-1}}} \tag{67}$$

$$\frac{V-U}{2}(T) = \sqrt{(\frac{V+U}{2})^2 + (r-1)e^r} \tag{68}$$

where r must be large enough that $\frac{V-U}{2}$ is real.

The ξ coordinate constant surfaces are given by

$$\frac{V-U}{2}(\xi) = \frac{\xi e^{\frac{r}{2}}}{\sqrt{r} + \frac{\text{asinh}(\sqrt{\text{r}-1})}{\sqrt{\text{r}-1}}} \tag{69}$$

$$\frac{V+U}{2}(\xi) = \pm\sqrt{(\frac{(V-U)}{2})^2 - (r-1)e^r} \tag{70}$$

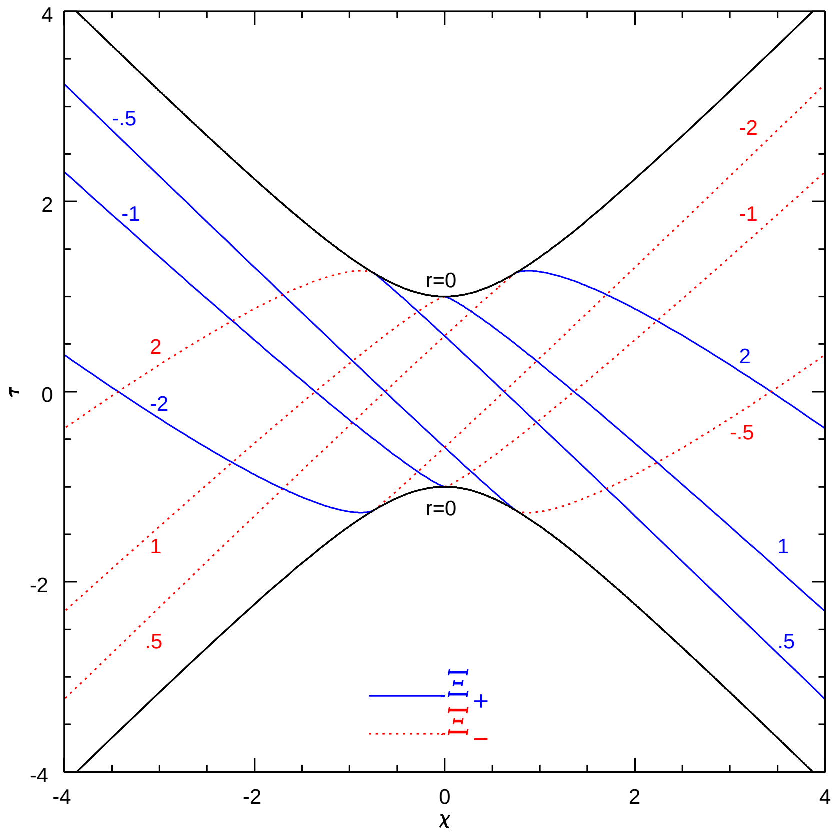

FIG. 1: The Ξ constant coordinate surfaces in the Kruskal coordinates. Each of those surfaces is a flat spatial slice. All begin at the r=0 singularity and go out to infinity. Note that both the Ξ_+ and the Ξ_- constant surfaces are spatial surfaces.

where the parameter r is chosen small enough so that $\frac{V+U}{2}(\xi)$ is real.

In figure 2 we have the plot of the T and ξ constant surfaces of the Synge metric plotted in the SK coordinate system.

The Lemaître coordinates are interesting because they look, at first, as though they are regular coordinates already which cover the whole spacetime. The metric

$$ds^2 = d\tau^2 - \frac{1}{r(\sigma - t)}d\sigma^2 - r(\sigma - t)^2(d\theta^2 - \sin(\theta)^2 d\phi^2) \tag{71}$$

looks regular everywhere.except at $r = 0$ of $t = \sigma$. But if we look at the null geodesics

$$\frac{d\sigma}{d\tau} = \pm\sqrt{(}r(\tau - \sigma)) = \pm(\frac{3}{2}(\sigma - \tau))^{\frac{1}{3}} \tag{72}$$

we find for the + sign, taking $z = \sigma - \tau$ that

$$\frac{dz}{d\tau} = \pm(\frac{3}{2}z)^{\frac{1}{3}} - 1 \tag{73}$$

The RHS goes to 0 when $z = \frac{2}{3}$ and τ goes to ∞ if we take the + sign in the equation for z. Ie, the null geodesics coming out of the black hole come from $\tau \to -\infty$. Had one taken the other solution (with $\tau \to -\tau$) for the Lemaître metric, it would be the ingoing null geodesics which would have terminated at $r = 1$. Ie, again the Lemaître coordinates cover only a part of the complete spacetime. The extended Lemaître coordinates ($\Sigma_\pm$) do cover the whole of the spacetime.

From the two graphs, of the extended PG coordinates, and the extended Lemaître coordinates, we can see the problem with the original Lemaître coordinates. The latter are essentially using the Ξ_- and the Σ_- coordinates. the

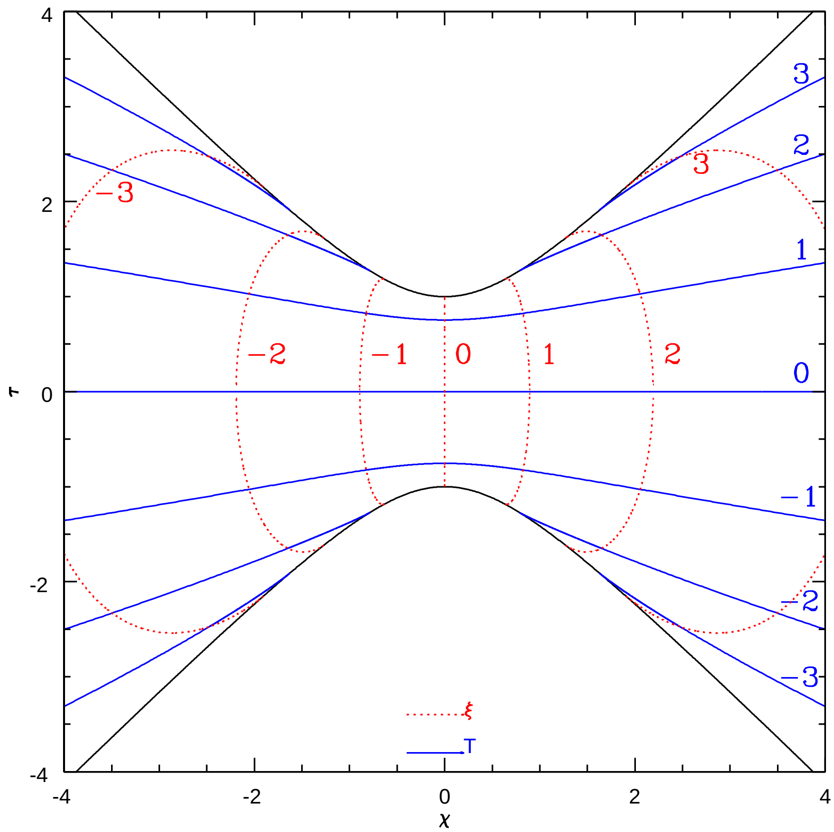

FIG. 2: The T, ξ constant Synge coordinate surfaces plotted in Kruskal coordinates. Note that while the T constant hypersurfaces are spacelike hypersurfaces, the ξ constant ones are not everywhere timelike. In particular near and within the horizon these surface become spacelike for large enough values of ξ.

problem with these is they become degenerate along the past horizon, where both are equal to zero. Ie, these (and the original Lemaître coordinates which are the logarithm of these coordinates) coordinates do not cover the past horizon. However, if we choose for example the Σ_+ and the Ξ_- coordinates, these do cover the whole of the extended spacetime, with no degeneracies. We have

$$\Sigma_+\Xi_- = \frac{\sqrt{r}-1}{\sqrt{r}+1} e^{\sqrt{r}(\frac{1}{3}r+1)} \tag{74}$$

$$\frac{\Sigma_+}{\Xi_-} = e^t e^{\frac{1}{3}r\sqrt{r}} \tag{75}$$

or

$$\frac{\sqrt{r}(r+2)}{2(r-1)}dr = \frac{d\Sigma_+}{\Sigma_+} + \frac{d\Xi_-}{\Xi_-} \tag{76}$$

$$dt + \frac{1}{2}\sqrt{r}dr = \frac{d\Sigma_+}{\Sigma_+} - \frac{d\Xi_-}{\Xi_-} \tag{77}$$

to give

$$ds^2 = \frac{\sqrt{r}+1}{(r+1)^2}\left[(\sqrt{r}+1)e^{-(r/3+2)\sqrt{r}}(\Xi_-^2 d\Sigma_+^2 - \Sigma_+^2 d\Xi_-^2) + 4d\Xi_- d\Sigma_+\right] + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2 \tag{78}$$

This shares with the original Lemaître coordinates that each of the Σ constant hypersurfaces have flat three dimensional spatial metrics, while each of the Ξ, θ, ϕ constant lines are timelike geodesics which have zero velocity at infinity. Unlike

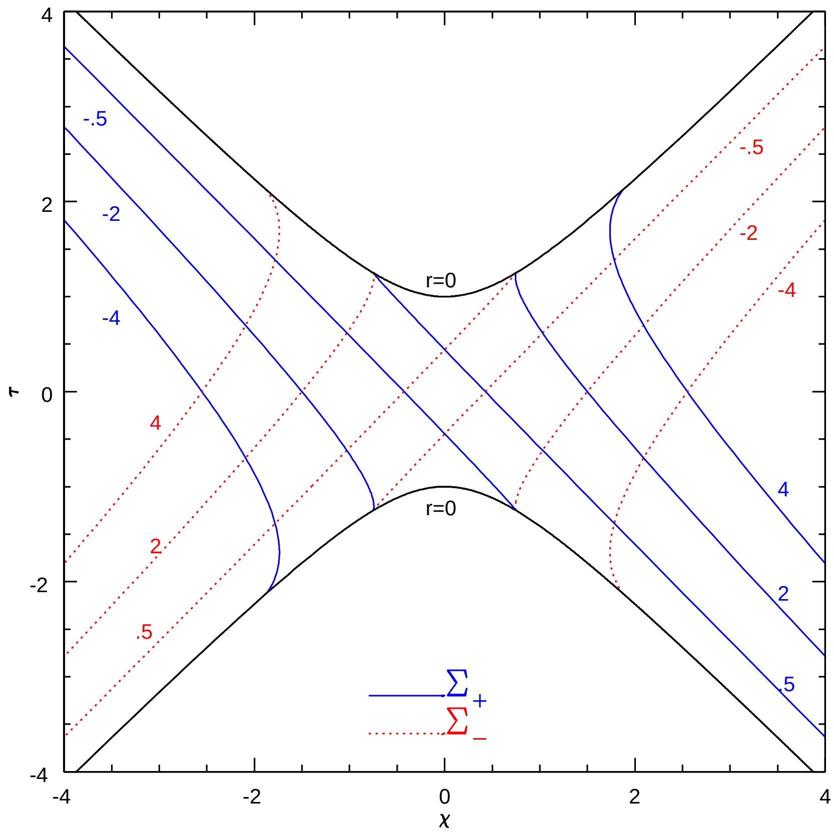

FIG. 3: The Lemaitre extended coordinates Ξ_+, Ξ_- plotted on the SK extended coordinates.

the original Lemaître coordinates however, they cover the whole of the analytic extension of Schwarzschild spacetime. They are thus just as simply married to the flat Robertson Walker dust universe model as were the original Lemaitre coordinates.

Finally, using the Fronsdal coordinates Θ, Y we plot the Θ constant and Y constant hypersurfaces. Note that these Θ constant hypersurfaces surfaces do not run into the $r = 0$ singularity. On the other hand, all of the Y constant lines originate at $T = \pm 1$, $\xi = 0$ points on the singularity, with the Y constant lines only being timelike for certain values of $Y < 2$ and only for certain values of Θ. Ie, the Y constant coordinate in these "Fronsdal" coordinates is very badly behaved near the $r = 0$ singularity while the Θ const. coordinate surfaces are nicely behaved.

In figure 5, we have the Israel-Newman coordinates plotted. The U coordinate is identical to the U coordinate of the SK metric, but the z coordinate is spacelike coordinate except along the future horizon, where it is null

Finally, in figure 6 we have the global coordinates obtained by taking the original "advanced and retarded" Eddington-Finkelstein time coordinates and exponentiating them to make the horizon regular. These for a pair of spacelike hypersurfaces (except again along the horizons where these coordinates become null.)

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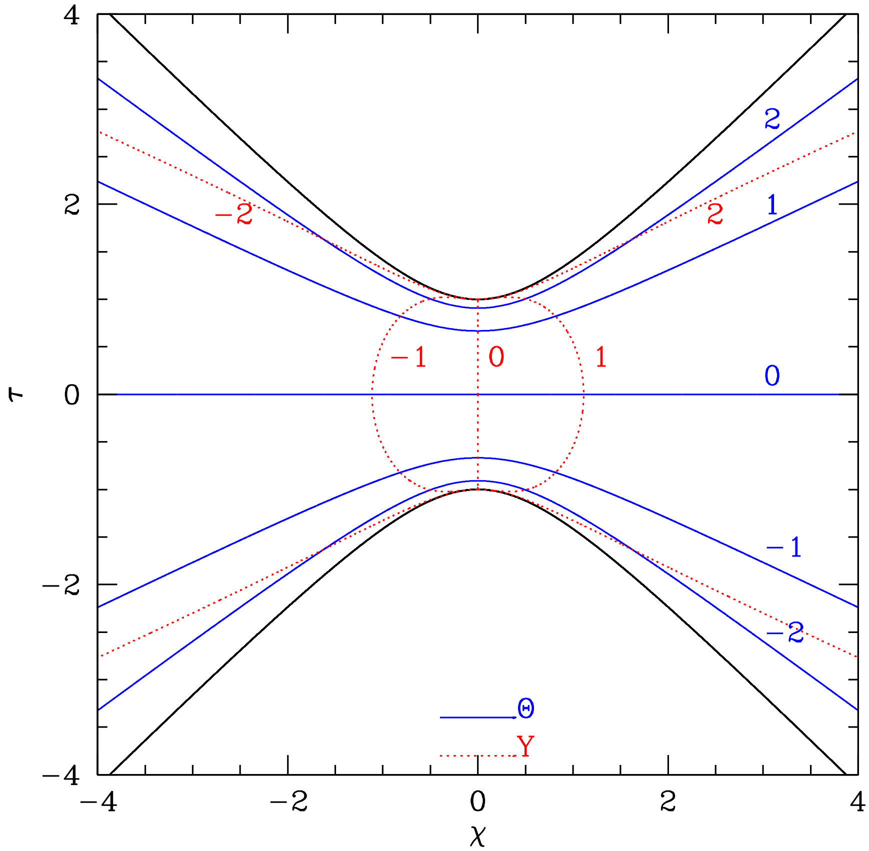

FIG. 4: The Θ and Y constant hypersurfaces for the Fronsdal embedding of Schwarzschild into a flat 6 dimensional spacetime, While the Y constant coordinates seem to hit the $r = 0$ singularity are various points, those surfaces actually skirt (as spacelike surfaces) extremely close to the singularity before finally all hitting it at the same point.

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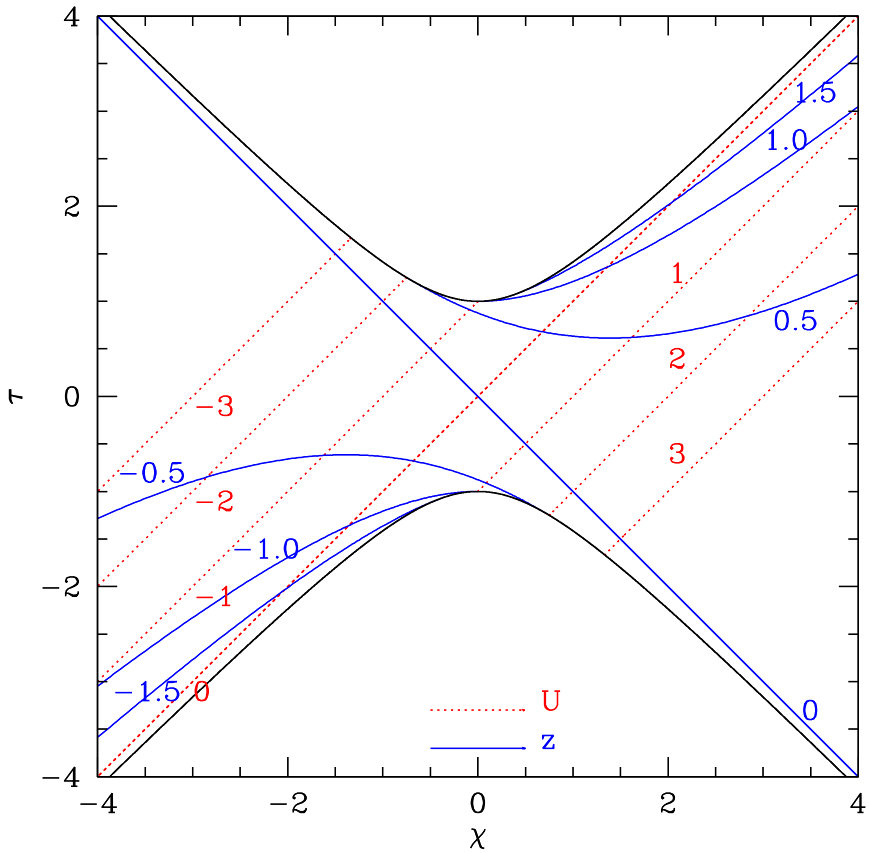

FIG. 5: The global U and z coordinates defined by the Israel-Newman metric coordinates. The z constant hypersurfaces are spacelike while the constant U slices are the same as for the SK coordinates..

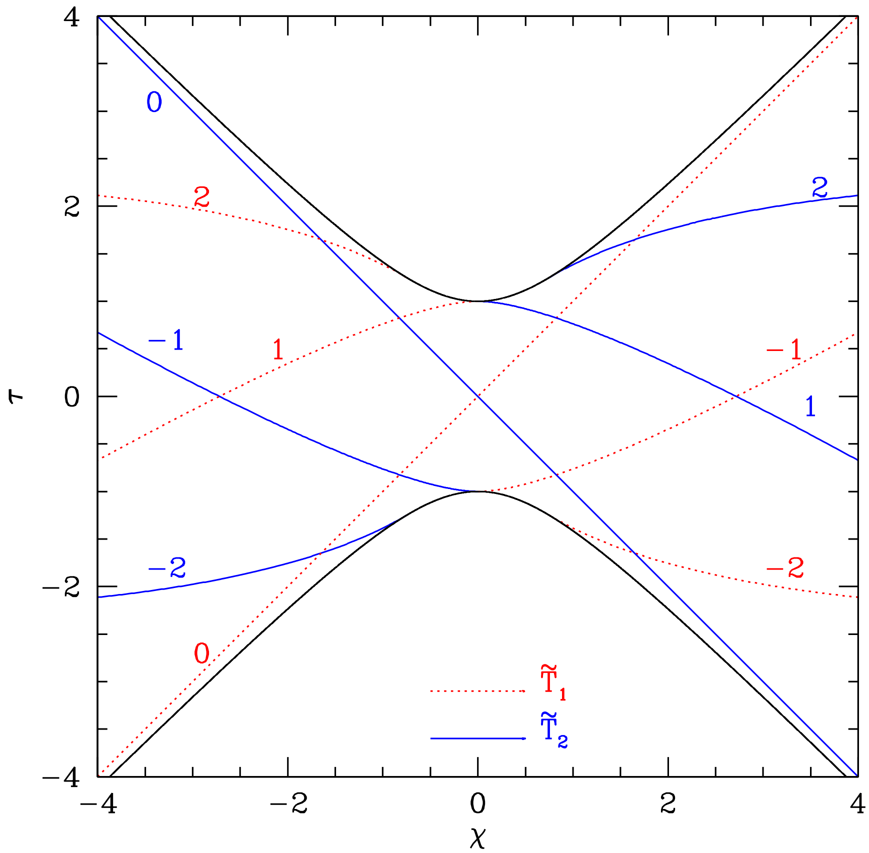

FIG. 6: The $\tilde{T}_1$ and $\tilde{T}_2$ coordinates, which are the exponentials of the advanced and retarded times as defined in the original Eddington and Finkelstein papers. These were non-null timelike coordinates. As with the IN coordinates, one of their advantages is that the metric and the coordinate transformations can be written as explicit functions of the coordinates, unlike the Kruscal coordinates for which the metric in the U, V coordinates is an implicit, parametric function of the coordinates.