General Relativitiy
Field theory

Quantum Field Theory

The application of the previous theory to field theory is straightforward. Instead of the index i, one takes the index to be the spatial point x and teh sum converts to an integral over x. The Lagrangian, in flat spacetime is

$$L = \frac{1}{2} \int (\partial_t \phi(t,x))^2 - (\nabla\phi \cdot \nabla\phi) - m^2 \phi) d^3x \tag{1}$$

with the momentum being

$$\pi(t,x) = \partial_t \phi \tag{2}$$

and the Hamiltonian

$$H = \frac{1}{2} \int (\pi^2 + (\nabla\phi \cdot \nabla\phi) + m^2 \phi^2) d^3x \tag{3}$$

The inner product between two complex solutions to the equations are

$$< \tilde{\phi}, \phi >= i \int (\tilde{\pi}(t,x)^* \phi(t,x) - \tilde{\phi}(t,x)^* \pi(t,x)) d^3x \tag{4}$$

$$= i \int (\dot{\tilde{\phi}(t,x)}^* \phi(t,x) - \tilde{\phi}(t,x)^* \phi(t,x)) d^3x \tag{5}$$

One can choose some arbitrary set of complex solutions, indexed by α namely $\phi_\alpha(t,x)$ which we demand that

$$< \phi_\alpha, \phi_\beta >= \delta_{\alpha\beta} \tag{6}$$

$$< \phi^*_\alpha, \phi_\beta >= 0 \tag{7}$$

Then we can define the annihilation operators

$$A_\alpha =< \phi_\alpha, \Phi > \tag{8}$$

where Π, Φ are the quantum operators which obey the field equations. Then we can write

$$\Phi = \sum_\alpha (A_\alpha \phi_\alpha(t,x) - A^\dagger_\alpha \phi^*(t,x)) \tag{9}$$

The Hamiltonian diagonalization is given by

$$\partial_t \phi_\omega = i\omega\phi_\omega \tag{10}$$

Choose the modes

$$\phi(t,x) = \phi_k \frac{e^{-ikx}}{\sqrt{(2\pi)^3}} \tag{11}$$

and similarly for π. The equations of motion are

$$\dot{\phi}_k = \pi_k \tag{12}$$

$$\pi_k = -(k^2 + m^2)\phi_k \tag{13}$$

which gives

$$\omega = \sqrt{k^2 + m^2} \tag{14}$$

The annihilation operator

$$A_{\omega k} = i \int \pi_k^* \Phi(t,x) - \phi_k^* \Pi(t,x) \frac{e^{ikx}}{\sqrt{(2\pi)^3}} d^3x \tag{15}$$

The annihilation operators $A_{\omega,k}$ also minimize the energy, and the state annihilated $A_{\omega k}|0\rangle$ is the usual vacuum.

Let us now choose a more complex situation. Write the Lagrangian as

$$L = \frac{1}{2} \int a(t)^3((\partial_t\phi)^2 - \frac{1}{a(t)^2}(\nabla\phi)^2 - m^2\phi^2) d^3x \tag{16}$$

This is just

$$\frac{1}{2} \int \sqrt{|g|} g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) d^3x \tag{17}$$

the coordinate invariant Lagrangian for the scalar field in the cosmological metric

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) \tag{18}$$

The conjugate momentum will be

$$\pi = a^3 \dot{\phi} \tag{19}$$

to give a Hamiltonian

$$H = \frac{1}{2} \int (\frac{\pi^2}{a^3} + a(\nabla\phi)^2 + a^3 m^2 \phi^2) d^3x \tag{20}$$

The inner product is again

$$< \phi, \tilde{\phi} >= i \int (\pi^* \tilde{\phi} - \phi^* \tilde{\pi}) d^3x \tag{21}$$

Choose the modes

$$\phi = \phi_k \frac{e^{-ikx}}{\sqrt{(2\pi)^3}} \tag{22}$$

and similarly for π. The Hamiltonian diagonalization at time t gives the eigenvectors and eigenvalues

$$i\omega\pi_{\omega k} = -(ak^2 + a^3 m^2)\phi_{\omega k} \tag{23}$$

$$i\omega\phi_{\omega k} = \frac{\pi_{\omega k}}{a^3} \tag{24}$$

$$\omega = \sqrt{\frac{k^2}{a^2} + m^2} \tag{25}$$

ω depends on time. These modes are not solutions of the equations of motion unless a is independent of time. However we can use the modes at the time t at which they are defined as above as inital data for a complete solution. In order to normalise the mode, we want

$$\delta(k, k') = i \int (\pi_k^* \phi_k' - \phi_k^* \pi_k') \int \frac{e^{i(k-k')x}}{(2\pi)^3} d^3x \tag{26}$$

$$= 2\omega a^3 |\phi_k|^2 \delta(k - k') \tag{27}$$

or

$$\phi_k = \frac{1}{\sqrt{2\omega a^3}} \tag{28}$$

The equations of motion in the Heisenberg representation can be written for $\Phi_k(t) = \int \frac{1}{\sqrt{(2\pi)^3}} e^{ikx}\Phi(t,x)$ and similarly for $\Pi_k(t)$. At time $t = t_0$ we have the equation

$$\dot{\Phi}_k(t) = \frac{1}{a(t_0)^3} Pi_k(t_0) \tag{29}$$

$$\dot{\Pi}_k(t) = -(a(t_0)k^2 + a^3(t_0))\Phi_k(t_0) = -\omega^2 a^3(t)\Phi_k \tag{30}$$

Thus we can define the Annihilation operators corresponding to this Hamiltonian diagonalisation at each time t.

$$A_k(t) < \phi_k \frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}, \Phi >= \sqrt{\frac{a^3\omega}{2}}\Phi_k - \frac{i}{\sqrt{2\omega a^3}}\Pi_k \tag{31}$$

where

$$\Phi_k = \int \Phi \frac{e^{ikx}}{\sqrt{(2\pi)^3}} d^3x \tag{32}$$

and similarly for Π_k.

Then

$$\frac{dA_k}{dt} = \sqrt{\frac{a^3\omega}{2}}\dot{\Phi}_k - \frac{i}{\sqrt{2\omega a^3}}\dot{\Pi}_k + \partial_t \ln(\sqrt{a^3\omega})(A^\dagger) \tag{33}$$

$$= i\omega A_k + \frac{\dot{a}}{a}(3 - \frac{k^2}{k^2 + m^2a^2})A_k^\dagger \tag{34}$$

and the Hamiltonian diagonaisation annihilation operator at time $t + \delta t$ is a mixture of the annihilation operators at time t. If one starts in the vacuum state at time t or its annihilation operators, it is not the vacuum state at time $t + \delta t$ but is rather a squeezed state. Defining the the number of particles at time $t + \delta t$ by the operator $N(t + \delta t) = A_k^\dagger(t + \delta t)A_k(t_\delta t)$ where the state is the vacuum state with respect to $A_k(t)|0_t\rangle = 0$

$$\langle 0_t|N(t + \delta t)|0_t\rangle \approx \delta t^2 \left(\frac{\dot{a}}{a}(3 - \frac{k^2}{k^2 + m^2a^2})\right)^2 \tag{35}$$

As $k \to \infty$, this approaches a term independent of k and depenent on $a(t)$ and its derivatives . When integrated over all $\vec{k}$ up to length K this diverges as $K^3\delta t^2$. If one defines ones particles by Hamiltonian diagonalization, then after the smallest instant of time, a vacuum state, a no-particle state, is converted into one with and infinite number of particles in it. (and an infinite energy compared to the minimum energy), most of those occuring at ultra large $\vec{K}$. which is clearly a wrong result.

This problem was recognized by Parker in the late 60's to early 70's. It led, and still leads, to huge controversies as to what is meant by particles or excitations in such a cosmological spacetime.

One approach is to change the definition of time. If we define a new time with

$$\tau = \int \frac{dt}{a(t)} \tag{36}$$

the so called conformal time, the Hamiltonian becomes

$$H_\tau = \frac{\Pi^2}{a^2} + a^2(\nabla\Phi)^2 + m^2a^4\Phi^2. \tag{37}$$

Now, define new field and conjugate momenta by

$$\Phi = \frac{\hat{\Phi}}{a} \tag{38}$$

$$\Pi = a\hat{\Pi} - \partial_t(a)\hat{\Phi} \tag{39}$$

The new Hamiltonian action, which gives the same equations of motion for Φ as the old Hamiltonian did becomes

$$\hat{H} = \frac{1}{2}(\hat{\Pi}^2 + (\nabla\hat{\Phi})^2 + (m^2a^2 - \frac{\partial_\tau^2 a}{a} - (\frac{\partial_\tau^2 a}{a})^2)\hat{\Phi}^2 \tag{40}$$

A direct calculation shows that the equations of motion given by this Hamiltonian in terms of the time τ are the same as before. Defining $h = \frac{\partial_\tau a}{a}$, we can calculate the Hamiltonian diagonalisation for this Hamiltonian from which we find

$$\hat{\omega} = \sqrt{k^2 + m^2a^2 - h^2 - \partial_\tau h} \tag{41}$$

and the normalisation factor for the solutions are

$$\phi_k = \frac{1}{\sqrt{2\hat{\omega}}} \tag{42}$$

which gives

$$\hat{A}_k(\tau) = \frac{1}{\sqrt{2}}(\sqrt{\hat{\omega}}\hat{\Phi}_k - \frac{i}{\sqrt{\hat{\omega}}}\hat{\Pi}_k) \tag{43}$$

and

$$\partial_\tau A_k = i\hat{\omega}A_k + \frac{\partial_\tau\hat{\omega}^2}{4\hat{\omega}^2}A_k^\dagger \tag{44}$$

The number of particles after time $\delta\tau$ then becomes

$$\langle\hat{0}_\tau|\hat{A}_k^\dagger(\tau+\delta\tau)\hat{A}_k(\tau+\delta\tau)|\hat{0}_\tau\rangle \approx O\left(\frac{\delta\tau^2}{k^4}\right) \tag{45}$$

and the total expectation value of the total number of particles goes as $\int \frac{k^2 dk}{k^4}$ for large k. This is finite as long as m is large enough . Ie, by simply redefining the time, and by redefining the field strength and the conjugate momentum, one can turn an infinite number of particles into a finite number. Does this mean that this finite number is the true particle creation rate? The answer is again no, because this transformation really has no physics behind it. It is an arbitrary mathematical manipulation, a re-coordinateization of phase space (from $P_k,\ Q_k$ to $\hat{P}_k,\ \hat{Q}_k$). Note also that if $ma(\tau)$ is small enough and the factor approriate, $\hat{\omega}$ can become imaginary for small k and small $a(\tau)$. One could make further redefinitions of the momentum and configuration variables, dependent for example on k, so as to make the convergence at large k even faster or to have have the ˆ phase space variables approach the original phase space variables for small k . Thus the particle creation number becomes a somewhat meaningless concept.

0.1 Appendix

Changing variables in Hamiltonian

In changing variables, one has two "coordinate" choices. One is a space-time coordinate choice (and in particular the time variable), and the other is cannonical transformation (ie a transformation of the vaiables one uses in phase space– what the p and q are). Both transformations can significantly alter the both the Hamiltonian, and the naive definiton of Hamiltonian diagonalisation.

The Hamiltonian action is

$$S = \int (\sum_i p_i \partial_t q_i - H(p,q))dt \tag{46}$$

To change configuration or momentum variables, and keep the solutions of the equations of motion equivalent, we must preserve the form of this action. Thus, for example if we change q_i to $q_i = \hat{q}_i \alpha\alpha$ we must also change p_i so $p^i = \frac{1}{\alpha}\hat{p}_i$. Ie, one must preserve the first term in the action, the $p_i\partial_t q_i$ terms, so that the new phase space veriables have the same form .

If α depends on time, we can define

$$(p_i \partial_t q_i)dt = \frac{1}{\alpha}\hat{p}_i \partial_t(\alpha \hat{q}_i) = \hat{p}_i \partial_t \hat{q}_i + \partial_t(\ln(\alpha))\hat{p}_i \hat{q}_i \tag{47}$$

The last term must be incorporated into the Hamiltonian and is a function of just $\hat{p}_i$ and $\hat{q}_i$. Thus we have

$$\hat{H}(\hat{p},\hat{q}) = H(\alpha\hat{p}, \frac{q}{\alpha}) + \partial_t(\ln(\alpha))\hat{p}_i\hat{q}_i \tag{48}$$

Another transformation one can make is to let

$$\tilde{p}_i = p_i + \beta q_i \tag{49}$$

$$\tilde{q}_i = q_i \tag{50}$$

and we get

$$p_i \partial_t q_i = (\tilde{p}_i - \beta q_i)\partial_t \tilde{q}_i = \tilde{p}_i \partial_t \tilde{q}_i + \frac{1}{2}(\partial_t \beta) q_i^2 - \partial_t(\beta q_i^2)) \tag{51}$$

The first term is just the usual term in the Hamiltonian action. The second term is one that should be incorporated into the Hamiltonian, and the third term is a complete derivative, and results in boundary terms in the action integral. If the variations of the action are to be taken so that δq is zero on the boundaries (necessary to get the usual Hamilton equations), then this third term will contribute nothing to the variation and can be eliminated. Thus we have

$$\tilde{H}(\tilde{p},\tilde{q}) = H(\tilde{p}-\beta\tilde{q},q) - \frac{1}{2}\partial_t\beta q_i^2) \tag{52}$$

For a single degree of freedom, one can continue in this fashion, using the second type of transformation to elimate cross terms (pq) in the Hamiltonian, (but at the price of introducing more complex configuration dependent potentials) and the first to get rid of dependence in the Mass terms or the potential terms, at the expense of introducing cross terms into the Hamiltonian. This alteration results in an assymptotic expansion. Eventually the approximation, where we assume that the time dependent terms are actually constant in time (Ie, the Hamiltonian diagonalisation) gets worse and worse instead of better. Which choice gives a true definition of particles? Why do we ever think of the field theory as if it could be described by particles? Only a thorough understanding of what the concept of particles is supposed to mean, physically, can answer such questions. And thus these are questions which bedeviled (and still bedevil) the field.