Physics 530-23
Cosmology

ne of the surprizing consequences of General Relativity was the existance of a field called Cosmology. Within Newtonian physics, the universe as a whole was the same everywhere and always. It was just matter that ran around within the universe. However, with the metric having dynamics it was possible for the universe as a whole to have change, have a history. Einsein at first strongly resisted this, and when Friedman pointed this out to him, he ignored it and then actively tried to eliminate it.

The simplifying assumption that Friedman made was that somehow in the large, the usnivers was the same everywhere and in all direction. It was homogeneous and isotropic, at least in space. It had the maxiximum number of Killing vectors in space– and since space is 3-D, that amounts to 6. (As with 4-D, where there are three independent initial vectors that the Killing vector could have, and three independent anti-symmetric tensors that the intial conditions for $\partial_{[\nu}\xi_{\mu]}$ at a point, one has a maximum of 6 possible Killing vectors. For example if the space was flat, there would be 3 translations, and three rotations possible. There are three different types of spatial metric one could have. As metioned the first is flat space. The second is spherical– ie the surface of three dimensional sphere. Thinking in 4-D we could look at $w^2 + x^2 + y^2 + z^2 = 1$ as a three dimensional sphere in a 4-D Euclidian space. Taking $r^2 = x^2 + y^2 + z^2$, this becomes $w^2 + r^2 = 1$ and the 4-D flat metric $dw^2 + dr^2 + r^2(d\theta^2 + sin(\theta)^2)$. But the constraint means that $wdw + rdr = 0$ so the above metric constrained to the 3 dimension spere becomes

$$dw^2 + dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{1}$$

$$= \frac{r^2}{w^2}dr^2 + dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2)) \tag{2}$$

$$= (\frac{r^2}{1-r^2} + 1)dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2)) \tag{3}$$

$$= \frac{1}{1-r^2}dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2)) \tag{4}$$

is another maximally symmetric 3-D metric. Or, defining $r = sin(\rho)$ this becomes

$$d\rho^2 + \sin(\rho)^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{5}$$

The symmetries of a sphere in the 4-D space are rotations in the $w-x,\ w-y;\ w-z;\ x-y;\ x-z; y-z$ planes.

However another possibility is to choose the 4-D flat space to be Minkowskian, with

$$w^2 - x^2 - y^2 - z^2 = 1 \tag{6}$$

or $w^2 - r^2 = 1$ which is a spacelike hyperbola. Then we get

$$wdw = rdr \tag{7}$$

$$dw^2 - dx^2 - dy^2 - dz^2 = \frac{r^2dr^2}{w^2} - dr^2 - r^2(d\theta^2 + sin(\theta)^2d\phi^2) \tag{8}$$

$$= -\frac{dr^2}{(1+r^2)} - r^2(d\theta^2 + sin(\theta)^2d\phi^2) \tag{9}$$

Or taking $r = \sinh(\zeta)$

$$d\zeta^2 + \sinh(\zeta)^2(d\theta^2 + sin(\theta)^2d\phi^2) \tag{10}$$

The symmetries of the relativistic hyperbola are the Lorentz boosts in the $w-x;\ w-y;\ w-z$ planes, and rotations in the $x-y;\ x-z;\ y-z$ planes. Again 6 independent symmetries.

To maintain the symmetries, we can choose coordinates such that the $g_{tr},\ g_{t\theta},\ g_{t\phi}$ are non-zero or you would ruin the symmtries. Furthermore, the components can only depend on the t coordinate. One thus gets the three types of metric,

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\omega^2 + \sin(\omega)^2d\phi^2)\right) \tag{11}$$

where $k = \{+1,\ 0,\ -1\}$ give the three alternatives, spherical, flat or hyperbolic three geometries.

In the following an important variable is

$$H = \partial_t ln(a(t)) = \frac{\dot{a}}{a} \tag{12}$$

which is called the "Hubble Constant". It is clearly not a constant unless $a(t) = e^{\pm H_0 t}$.

We now need to calculate Einstein Tensor. Lets take the simplest one, the flat metric where we can write it as

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) \tag{13}$$

Since the only time dependent terms are $g_{xx},\ g_{yy},\ g_z z$, the only Christoffel symbols are

$$\Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz} = \dot{a} = a(t)H(t) \tag{14}$$

$$\Gamma^x{}_{tx} = \Gamma^y{}_{ty} = \Gamma^z{}_{tz} = \frac{\dot{a}}{a} = H(t) \tag{15}$$

$$R^\mu{}_{\nu\rho\sigma} = \partial_{[\rho|}\Gamma^\mu{}_{\nu|\sigma]} + \Gamma^\mu{}_{\alpha[\rho}\Gamma^\alpha{}_{\sigma]\nu} \tag{16}$$

where we recall that the [] means antisymmetrization.

Going through the algebra, we find

$$G_{tt} = 3H^2 \tag{17}$$

$$G_{xx} = G_{yy} = G_{zz} = -a^2(2\dot{H} + 3H^2) = g_{xx}(2\dot{H} + 3H^2) \tag{18}$$

or

$$G_{ab} = (2\dot{H} + 3H^2)g_{ab} \tag{19}$$

where $a,\ b$ takes values $1, 2, 3$

If we take a perfect stationary fluid (it would have to be stationary to satisfy the symmetries we want)where

$$T_{tt} = \rho \qquad \text{The energy density} \tag{20}$$

$$T_{ab} = pg_{ab} \qquad \text{The pressure} \tag{21}$$

The trace will be $T = \rho - 3p$. For massless radiation (eg, photons) we will have $p = \frac{1}{3}\rho$ and the energy momentum tensor is traceless. For dust, the pressure is 0. For the "cosmological constant" $p = -\rho$

The equations of motion are

$$3H^2 = 8\pi G\rho \tag{22}$$

$$(2\dot{H} + 3H^2) = -8\pi Gp \tag{23}$$

Note that $\partial_t(a^3 3H^2) = a^3(9H^3 + 6H\dot{H}) = 3a^3 H(2\dot{H} + 3H^2)$ which is the Bianci identity. This gives $\partial_t(a^3\rho) = -3a^3 Hp$ Since $a^3 d^3x$ is the volume element, this gives

$$\partial_t(\rho V) + p\partial_t V = 0 \tag{24}$$

which is just the conservation of energy equation for a fluid. If $p = \alpha\rho$, we get

$$\partial_t ln(\rho) = -(1+\alpha)\partial_t ln(a^3) \tag{25}$$

$$\rho a^{(1+\alpha)/3} = \text{const} \tag{26}$$

Thus for dust the density falls as $1/a^3$, for massless radiation as $1/a^4$ and for "dark energy", ρ is constant. Finally for $\alpha = -\frac{1}{3}$, the density falls as $\frac{1}{a^2}$.

For generic k, we have

$$G_{tt} = 3(H^2 - k/a^2) \tag{27}$$

$$G_{ab} = g_{ab}\left(2\ddot{a}/a + H^2 - k/a^2\right) \tag{28}$$

Ie, the spatial "curvature" acts as if it were matter with an equation of state of $alpha = -\frac{1}{3}$. If $k = 1$ then the effective energy density is positive. If it is $k = -1$ then the effective energy density is negative.

If the equation of state is $p = -\rho$ which is a constant, then the result is known as DeSitter space and the energy-momentum tensor is just proportional to the 4-D metric.

Note that this also implies that the scale factor obeys

$$3H^2 = 8\pi G\rho = 8\pi G\rho_0/a^{(1+\alpha)/3} \tag{29}$$

which gives

$$a \propto t^{-2/(3(1+\alpha)} \tag{30}$$