Physics 530
Tensors

Metric and variations. One of the techniques that had grown during the 19th century was that of the calculus of variations. The idea was that one could write down an action for the system, a n integral over a function (like a Lagangian or a Hamiltonian action— differeing in that the second was a Lagrangian in which the action was linear in first derivatives with respect to time of the dynamic variables of the theory.) The idea was that the equations of motion were those functions of time and of space which extremised the action. Ie, if we designate the variables with $\psi(t,x)$ where x could be coordinates in a D dimensional space, or could be indices which were summed over. The consider a solution of the equations of motion, $\psi_0(t,x)$ which is an extremum of the action. Ie, consider a set of funtions $\psi_\epsilon(t,x)$ with $\psi_\epsilon(t,x)$ such that

$$I(\epsilon) = \int S(\psi_\epsilon(t,x))dtd^3x \qquad (1)$$

then we want that $\frac{dI}{d\epsilon}|_{\epsilon=0} = 0$for all function of ψ_ϵ. One may want to restict the functions of ϵ according to their values on the boundaries of the spacetime, so that they integral does not depend on those boundry values.

In GR, one wants that the integral be independent of the coordinates that one chooses to express that integral. We therefor need that the integral be independent of the coordinate choice. The first thing is to require that dtd^xgive an integral over physical, not coordinate volumes.

Consider, in flat space, one has a set of vectors $l_0^i, l_1^i, l_2^i, l_3^i$ which are lineary indendent (Ie there are no numbers α_j such that $\sum_j \alpha_j l_j^i = 0$.Then the question is, what is the volume of the paralleapiped made up of the four vectors l_j^i? Length times width time height times duration, but this is not simply the product of the leghts of these vectors, but each of length, width, height and duration must be perpendicular to the all the others as well. Let us also assume that they are all unit vectors. Define

$$\tilde{l}_0^i = l_0^i \qquad (2)$$

$$\tilde{l}_1^i = l_1^i - \frac{l_1^m g_{mn}\tilde{l}_0^n\tilde{l}_0^i}{\tilde{l}_0^k g_{kl}\tilde{l}_0^l} \qquad (3)$$

$$\tilde{l}^i_2 = l^i_2 - \frac{\tilde{l}^m_0 g_{mn} l^n_2 \tilde{l}^i_0}{\tilde{l}^k_0 g_{kl} \tilde{l}^l_0} - \frac{\tilde{l}^m_1 g_{mn} l^n_1 \tilde{l}^i_0}{\tilde{l}^k_1 g_{kl} \tilde{l}^l_1} \quad (4)$$

$$\ldots \quad (5)$$

In each case the next ˜ is made by subtracting the inner product between each of the previous unit tilde vectors and the current vector times the previous tilde vector. Each of the ˜ vectors are then orthogoanl to each of the the other tilde vectors. The lengths then of these mutually orthogonal vectors will be $Length^2_j = \tilde{l}^i_k g_{ij} l^j_k$. and the volume squared of the parallelapipe will be $Length^2_0 Length^2_1 Length^2_2 Length^2_3$, Now, take l^i_j to be the vector parallel to the jth coordinate axis, and with parameter along the curve to be x^j and take the vector to be dx^j. Then the volume will be proportional to $dx^0 dx^1 dx^2 dx^3$ with the proportionalty being some product of the metric components. The value will be a product of components of the metric. This turns out to be the determinant of the metric in the volume squared, or the square root of the determinant in the volume. Thus, the physical (metric) volume is $\sqrt{det(g_{ij}}$. This expression is linear in one set of terms and does not appear at all in any of the other terms. The some of those other terms is called the minor of the term. Ie, if we look for say g_{23}, then it will be multipie with a term M^{23}. M^{23} will not contain g_{23}, nor will any of the remaining terms.The minor will not contain any the elements in either the row or column of the factor g_{23}. in question. The determinat will equal

$$g = g_{20} M^{20} + g_{21} M^{21} + g_{22} M^{22} + g_{23} M^{23} +$$

Since any matrix with duplicated rows or columns has a zero determinant, and since $\sum_k g_i k M^{jk}$ is the determinant in which the ith row is the same as the jth row, this sum is zero unless i=j. And if i=j, this gives the determinant. Thus we have

$$\sum_k g_{ik} M^{jk} = det(g) \delta^j_i \quad (6)$$

which means that $M^{jk} = det(g) g^{jk}$ Also $det(g^{jk}) det(g_{ik}) = 1$ since the determinant of a product of two matricies is the product of the determinats, so $g_{ij}/detg$ is the minor of g^{ij} in the determinant of g^{ij}.

If one makes g_{ij} be a function of ϵ then we can write

$$\frac{ddet(g)}{d\epsilon}|_{\epsilon=0} = det(g) g^{ij} \frac{dg_{ij}}{d\epsilon}$$

or as usually written

$$\delta detg = det(g)g^{ij}\delta g_{ij}$$

Also since $g^{ik}g_{jk} = \delta^i_k$ we also get

$$g^{ik}\delta g_{jk} + \delta g^{ik}g_{jk} = 0 \tag{7}$$

or

$$\delta g^{ik} = -g^{ik}\delta g_{lk}g^{il}$$

or

$$\delta det(g) = -det(g)g_{ij}\delta g^{ij}$$

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0.1 Electromagnetism

Define the tensor

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{8}$$

where $A_0 = \phi$ the electromagnetic potential, and A_a are the components of the vector potential $\vec{A}$. (indices a, b, c, d, e, f go from 1 t0 3) Then, assuming Minkowski coordinates, F_{0a} are the components of the Electric field, $\vec{E}$ and $F_{ab} = \partial_a A_b - \partial_b A_a = B_c$ where a, b, c are an even permutation of $1, 2, 3$. source free Maxwell's equations then are

$$\partial_n u F_m u^\nu = 0 \tag{9}$$

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \tag{10}$$

These equations can be derived from the action

$$I = -\frac{1}{4}\int \sqrt{(}g)g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}d^4x \tag{11}$$

where $F_{\mu\nu}$ here is used as shorthand for $\partial_\mu A_\nu - \partial_\nu A_\mu$. For a Minkowski metric, the variation of this with respect to the δA_μ gives the Maxwell equations.

Note that

$$F^{\mu\nu}F_{\mu\nu} = F^{0a}F_{0a} + F^{a0}F_{a0} + F_{ab}F^{ab} = -2\vec{E}\cdot\vec{E} + 2\vec{B}\cdot\vec{B} \tag{12}$$

so, the action is

$$I = \frac{1}{2} \int (\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}) d^4x \tag{13}$$

In a guage where $\phi = 0 = A_0$, the first term is the Kinetic term $(\partial_t \vec{A} \cdot \partial_t \vec{A})$ while the second is like a potential term.

Let us now vary this with respect to δg^{munu}. This gives

$$\delta_{g^{\kappa\lambda}} I = -\frac{1}{4} \int (\sqrt{g} \left(-\frac{1}{2} g_{\kappa\lambda} \delta g^{\kappa\lambda} \right. \tag{14}$$

$$\left. + F^{\mu\nu} F_{\mu\nu} + (F_{\kappa\nu} g^{\nu\sigma} F_{\lambda}\sigma + F_{\mu\kappa} g^{\mu\rho} F_{\rho\lambda}) \delta g^{\kappa\lambda} \right) d^4x \tag{15}$$

The "kinetic energy" (time derivative of the fundamental fields, in this case $\partial_t A_a$ should have the form $\frac{1}{2} \sum_a (\partial_t A_a)^2) = \frac{1}{2} E \cdot E$, which suggests that $\sqrt{|g|} T_{\mu\nu} = 2 \frac{\delta I}{\delta g^{\mu\nu}}$. From the antisymmetry of F the last two terms are the same (changing the names of the dummy variables ν, σ to μ, ρ). This gives the energy Momentum tensor in flat spacetime

$$T_{\kappa\lambda} = (F_{\kappa\nu} F_{\lambda}{}^{\nu} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} g_{\kappa\lambda}) \tag{16}$$

(note that the Trace of this is 0 $g^{\kappa\lambda} T_{\kappa\lambda} = 0$).The $T_0 0$ term is given by

$$T_{00} = \vec{E} \cdot \vec{E} - \frac{1}{4} (2(\vec{E} \cdot \vec{E} - \vec{B} \vec{B})) \tag{17}$$

$$= \frac{1}{2} (\vec{E} \cdot \vec{E} + \vec{B} \vec{B}) \tag{18}$$

which is the usual expression for the energy of the electromagnetic field. The term

$$T_{oa} = -\frac{1}{2} F_{0b} g^{bd} F_{ad} = \frac{1}{2} E_b F_{ab} = \frac{1}{2} \vec{E} \times \vec{B} \tag{19}$$

is just the Poynting vector.

0.2 Massive scalar field

One can carry out the same kind of discussion with a scalar field

$$I = \frac{1}{2} \int \sqrt{(}g) (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} - m^2 \phi^2) d^4x \tag{20}$$

Again, varying twice this with respect to $g^{\kappa\lambda}$ we get

$$\delta I = \int \sqrt{g} \left(-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - m^2\phi^2)g_{\kappa\lambda}\delta g^{\kappa\lambda} \right. \tag{21}$$

$$\left. +\partial_\kappa\phi\partial_\lambda\phi\delta g^{\kappa\lambda} \right) d^4x \tag{22}$$

$$= \int \sqrt{|g|}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - m^2\phi^2)g_{\mu\nu})\delta g^{\mu\nu}d^4x) \tag{23}$$

0.3 Isentropic, Barytropic, irrotational perfect fluid

In this section I follow : S. Matarrese "On the Classical and Quantum Irrotational Motions of a Relativistic Perfect Fluid. I. Classical Theory" Proc Roy Soc A401, 53-66 (1985). Although he goes in the opposite direction, starting with the energy-momentum tensor, and deriving the necessary action. Note also that the assumptions are needed in order to ensure that the fluid is self contained and thermodynamically reversible. This ensures that energy and momentum are conserved and are not dissipated either by interactions with a heat bath or internally between various parts of the fluid (eg, internal friction).

Another possibility is the energy momentum tensor of an irrotational fluid, isentropic, barytropic perfect fluid. (barytropic means that the pressure is a function of the energy density ρ, isentropic means that the entropy per particle is constant). Define n as the particle number density, w as the enthalpy per particle, $\frac{\rho+p}{n}$, of the fluid, where n is the particle density.

Using the thermodynic relation for and an-isentropic (no heat flow between values) fluid $pdv = -d\epsilon$ where v is the volume per particle($\frac{1}{n}$), and ϵ the energy per particle. We have

$$pd\frac{1}{n} = -d\frac{\rho}{n} \tag{24}$$

which means that $dw = \frac{dp}{n}$ or $n = \frac{dp}{dw}$

We define the fluid as irrotational if $\partial_\mu(wu_\nu) - (\partial_\nu u_\mu) = 0$, where u^μ is the proper velocity of the fluid ($g^{\mu\nu}u_\mu u_\nu = 1$). This means that the we can define $wu_\mu = \partial_\mu\psi$ and $w^2 = \partial_\mu\psi\partial_\nu\psi g^{\mu\nu}$

Since, by definition, $w = \frac{p+\rho}{n} = \sqrt{\partial_\mu\psi\partial_\nu\psi g^{\mu\nu}}$ and since ρ is a function of p, we find that p is a function of derivatives of the scalar field, and the upper metric, i.e.,

$$w^2 = \partial_\mu\psi\partial_\nu\psi g^{\mu\nu} \tag{25}$$

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The action is

$$I = \frac{1}{2}\int \sqrt{|g|}p(w = \sqrt{\partial_\mu\psi\partial_\nu\psi g^{\mu\nu}})d^4x \tag{26}$$

which is a function only of $g^{\mu\nu}$ and of the derivatives of ϕ. The equation of motion of the fluid is then the variation of the above by ϕ, which would be

$$\delta I = \int \sqrt{|g|}\frac{n}{w}g^{\mu\nu}\partial_\nu\psi\partial_\nu\delta\psi d^4x \tag{27}$$

$$= \int \sqrt{|g|}(\frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{g}p'g^{\mu\nu}\partial_\mu\psi))\delta\psi d^4x \tag{28}$$

where p' is the derivative of p with respect to its argument. Thus we have

$$(\frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{g}p'g^{\mu\nu}\partial_\mu\psi)) = 0 \tag{29}$$

a highly non-linear differential equation for ψ since p is a highly non-linear founction of its argument.

The energy momentum tensor is then give by twice the variation of the action with respect to $g^{\mu\nu}$ divided by $\sqrt{|g|}$. This is

$$T_\mu\nu = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \tag{30}$$

which the equations of motion of the fluid ensure that it is conserved. Note that if the fluid is at rest ($u^a = 0$) then $T^tt = \rho$ the energy density of the fluid. The trace of the fluid $g^{\mu\nu}T_{\mu\nu}$ is then equal to $\rho + 3p$. For a fluid of relativistic particles (eg photons) the trace is equal to zero, and $p = \frac{1}{3}\rho$, which is not surprizing for photons, which are electromagnetic fields whose energy-momentum trace is equal to 0.