General Relativitiy
Lie derivative

Lie Derivative

In addition to the so called parallel or covariant derivative, there is also an additional concept called the Lie derivative. This derivative is more primative than the covariant derivative in that it assumes less structure on the spacetime.

Assume that we have a series of curves which fill the spacetime. Ie, through each point in the spacetime, there exists a curve from that series of curves, going through that point. We can now use these series of curves to slide the spacetime over itself and to slide any structures on the spacetime over itself. Let us designate the curve from this series of curves going through the point p to be designated by $\gamma_p(\lambda)$ and let the value of the parameter lambda designating the point p to be given by λ_p. Ie, $\gamma_p(\lambda_p) = p$. Now consider the point in the spacetime designated by $\gamma_p(\lambda_p + \mu)$. This will be a new point in the spacetime, near the point p. Let the tangent vector to this curve at p be $\frac{\partial}{\gamma_p}$.

Now consider a function $f(p)$. Define the Lie derivative of the function, designated by

$$\pounds_{\frac{\partial}{\partial\gamma_p}} f = \lim_{\epsilon\rightarrow 0} \frac{f(\gamma_p(\lambda_p+\epsilon)) - f(p)}{\epsilon} \tag{1}$$

We note that this is just the derivative of f along the curve γ_p and thus this is just

$$\pounds_{\frac{\partial}{\partial\gamma_p}} f = \frac{\partial}{\partial\gamma_p}^A (df)_A \tag{2}$$

or in coordinates,

$$\pounds_{\frac{\partial}{\partial\gamma_p}} \; f = \eta^i \partial_i f \tag{3}$$

where we define

$$\left(\frac{\partial}{\partial\gamma_p}\right)^A = \eta^i \left(\frac{\partial}{\partial x^i}\right)^A . \tag{4}$$

Now, let us consider the derivative of the cotangent vector defined by the function f. Ie, we want to define the derivative of the cotangent vector $\pounds_{\frac{\partial}{\partial\gamma_p}} df_A$. We do this by subtracting the cotangent vector defined by the dragged function

$$\tilde{f}_\epsilon(p) = f(\gamma_p(\lambda_p - \epsilon)). \tag{5}$$

where $\gamma_p(\lambda_p) = p$. Ie, γ_p is a curve which goes through the point p and λ_p is the value of the parameter when the curve is at the point p. Note that this assumes that we have a whole bunch of curves which go through every point in the neighbourhood of the point we are interested in, and p is an arbitrary point in that neigbourhood.

We now have the two cotangent vectors df_A and $(d\tilde{f}_\epsilon)_A$ defined at the point p. We can now define the derivative by

$$\pounds_{\left(\frac{\partial}{\partial\gamma_p}\right)^A} df_B = \lim_{\epsilon\to 0} \frac{df_B(p) - (d\tilde{f}_\epsilon(p))_B}{\epsilon} \tag{6}$$

Ie, we define this derivative by comparing the cotangent vector at the point p with that dragged to the point p by the action of the set of curves.

Writing this in coordinate form, we have

$$\tilde{f}_\epsilon(x^i(p)) = f(x^i(\gamma_p(\lambda_p - \epsilon))) \approx f(x^i) - \epsilon\eta^j\partial_j f + O(\epsilon^2) \tag{7}$$

The components of the cotangent vector are

$$(d\tilde{f}_\epsilon(p))_i = \partial_i(\tilde{f}_\epsilon(p)) = \partial_i f - \epsilon\eta^j\partial_j f) \tag{8}$$

and the Lie derivative then is

$$\pounds_{\eta^A} = \partial_i\eta^j\partial_j f + \eta^j\partial_j(\partial_i f) \tag{9}$$

Thus for a generic cotangent vector with components U_i we have

$$\pounds_{\eta^A} U_i = \eta^j\partial_j U_i + U_j\partial_i\eta^j \tag{10}$$

We can equivalently define the Lie derivative of a tangent vector by noting that $V^A W_A$ is an ordinary function, and thus

$$\pounds_{\eta^A} V^B W_B = \pounds_{\left(\frac{\partial}{\partial\gamma_p}\right)^A} V^i W_i \tag{11}$$

$$= (\eta^j \partial_j V^i) W_i + V^i (\eta^j \partial_j W_i) \tag{12}$$
$$= (\eta^j \partial_j V^i - V^j \partial_j \eta^i)(W_i) + V^i (\eta^j \partial_j W_i + W_j \partial_i \eta^j) \tag{13}$$
$$(\pounds_{\left(\frac{\partial}{\partial \gamma_p}\right)^A} W_B)_i V^i + W_i((\eta^j \partial_j V^i - V^j \partial_j \eta^i) \tag{14}$$

Thus we define

$$\pounds_{\eta^A} V^i = (\eta^j \partial_j V^i - V^j \partial_j \eta^i) \tag{15}$$

Note that

$$\pounds_{V^A} U^B + \pounds_{U^A} V^B = 0 \tag{16}$$

The Lie derivative of a tangent vector along another tangent vector is sometimes called the commutator of those two tangent vector fields.

It is very important to note that the Lie derivative is defined without any notion of metric and without any notion of covarient derivative. It is in many ways a more primative notion of derivative than is the covariant derivative. It requires fewer structures on the spacetime to be defined.

It also differs from the parallel derivative in that it is not linear in direction one is taking the derivative in (ie in the tangent vector to the curve) but also depends on the tangent vector to the nearby curves (it depends on derivatives of the tangent vector). There is no tensor (such as ∇_A) for the Lie derivative in some arbitrary direction. You must always designate the vector field along which you are taking the Lie derivative.

The Lie derivative of the metric is given by

$$\pounds_{\eta^A} g_{ij} = \eta^k \partial_k g_{ij} + g_{ik} \partial_j \eta^k + g_{kj} \partial_i \eta^k \tag{17}$$
$$= \eta^k \partial_k g_{ij} + \partial_j \eta_i + \partial_i \eta_j - \eta^k (\partial_i g_{kj} + \partial_j g_{ik}) \tag{18}$$
$$= \partial_j \eta_i + \partial_i \eta_j - 2\eta^k \Gamma_{kij} \tag{19}$$
$$= \partial_j \eta_i + \partial_i \eta_j - 2\eta_k \Gamma^k_{ij} \tag{20}$$

or

$$\pounds_V g_{AB} = \nabla_A V_B + \nabla_B V_A \tag{21}$$

Now, if the metric dragged along the curve is identical to the metric already there, and if this is true everywhere, then the geometry of the space dragged over itself is identical to the geometry of the space. this is called a symmetry of the spacetime. This means that if there exists a vector field K^A such that

$$\pounds_{K^A} g_{BC} = 0 \tag{22}$$

then the vector field K^A is a symmetry of the spacetime. Such vectors are called Killing vectors.

A 4-dimensional spacetime can contain at most 10 linearly independent Killing vectors.

Consider the Killing equation components

$$\partial_i K_j + \partial_j K_i = 2\Gamma^k_{ij} K_k \tag{23}$$

which says that the symmetric ordinary derivatives of the Killing vector can be written as a function of the Killing vector components themselves. The we can write the ordinary derivative of the Killing vector as

$$\partial_i K_j = \frac{1}{2}(\partial_i K_j - \partial_j K_i) \tag{24}$$

$$+\frac{1}{2}(\partial_i K_j + \partial_j K_i) \tag{25}$$

$$= \frac{1}{2}(\partial_i K_j - \partial_j K_i) + K_k \Gamma^k_{ij} \tag{26}$$

Ie the derivative of K^i in the direction j can be written in terms of the antisymmetric derivative of K and of the value of K at that point.

We can also write the Killing equation as

$$\partial_i K_j = -\partial_j K_i + 2K_k \Gamma^k_{ij} \tag{27}$$

Looking at the derivative of the antisymmetric derivative

$$\partial_k(\partial_i K_j - \partial_j K_i) = \partial_i(\partial_k K_j) - \partial_j \partial_k K_i \tag{28}$$

$$= -\partial_i \partial_j K_k + \partial_j \partial_i K_k + 2\partial_i(\Gamma^l_{kj} K_l) - \partial_j(\Gamma^l_{ik} K_l) \tag{29}$$

$$= 2(\partial_i \Gamma^l_{kj}) - 2(\partial_j \Gamma^l_{ki}) K_l + 2\Gamma^l_{ki} \partial_j K_l - 2\Gamma^l_{kj} \partial_i K_l \tag{30}$$

$$= 2\left(\partial_i \Gamma^l_{kj} - \partial_j \Gamma^l_{ki} + \Gamma^m_{ki} \Gamma^l_{jm} - \Gamma^m_{kj} \Gamma^l_{im}\right) K_l \tag{31}$$

$$+ \left(\Gamma^l_{ki}(\partial_j K_l - \partial_l K_j) - \Gamma^l_{kj}(\partial_i K_l - \partial_l K_i)\right) \tag{32}$$

Ie, the derivative of the antisymmetric derivative can be expressed in terms of derivatives of the metric times the components of the Killing vector plus derivatives of the metric times components of the antisymmetric derivative of the Killing tensor (since the ordinary derivative can be expressed in terms of the antisymmetric derivative and derivatives of the metric times the components of the Killing vector.). Ie, we have an intial value equation, in which if we specify the 4 components of the Killing vector and the six components of the antisymmetric derivative of the Killing vector at a point, then we can integrate them up along all of the coordinate axes, and everywhere in the spacetime.

It is of course also required that if we integrate up the equations along different paths, we get the same vector. This is what can reduce the number of Killing vectors to less than 10, but there can never be more than 10.

Flat spacetime has 10.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{33}$$
$$K(1)_i = (1, 0, 0, 0) \tag{34}$$
$$K(2)_i = (0, 1, 0, 0) \tag{35}$$
$$K(3)_i = (0, 0, 1, 0) \tag{36}$$
$$K(4) = (0, 0, 0, 1) \tag{37}$$
$$K(5)_i = (x, -t, 0, 0) \tag{38}$$
$$K(6)_i = (y, 0, -t, 0) \tag{39}$$
$$K(7)_i = (z, 0, 0, -t) \tag{40}$$
$$K(8)_i = (0, y, -x, 0) \tag{41}$$
$$K(9)_i = (0, z, 0, -x) \tag{42}$$
$$K(10)_i = (0, 0, z, -y) \tag{43}$$

where the (a, b, c, d) means that the t component is a, the x is b, the y is c and the z is d.

The first four have zero antisymmetric derivatives at $t = x = y = z = 0$, but have non-szero value for one of the components of the Killing vector at that point. The last 6 have zero value for all components at $t = x = y = z = 0$, but have non-zero antisymmetric derivative there.

Note that any linear combination of Killing vectors with constant coefficients is also a Killing vector. Similarly the Lie derivative of a Killing vector by another Killing vector is, assuming it is not 0, also a Killing vector.

[Since this last is non-trivial I will insert a derivation which however uses ideas from later in the lectures.)

One way to write Killing's equations is

$$K^A \nabla_A g_{BC} + g_{AC} \nabla_B K^A + g_{BA} \nabla_C K^C \tag{44}$$
$$= \nabla_B K_C + \nabla_C K_B = 0 \tag{45}$$

If $K^{1)}_A$ and$K^{2)}_A$ are by assumption two Killing vectors, we llok at the vector $K^{3)}_A = K^{1)A} \nabla_A K^{2)}_B - K^{2)A} \nabla_A K^{1)}_B$, the Lie derivative of the second with respect to the first. Then testing $K^{3)}_A$ to see if it is a Killing vector, we get $\nabla_B K^{3)}_C + \nabla_C K^{3)}_B$This has two terms

$$(\nabla_C K^{1)A}) \nabla_A K^{2)}_B - (\nabla_C K^{2)A}) \nabla_A K^{1)}_B \tag{46}$$
$$= g^{AD}((\nabla_C K^{1)}_D) \nabla_A K^{2)}_B - (\nabla_C K^{2)}_D) \nabla_A K^{1)}_) \tag{47}$$
$$= -g^{AD}((\nabla_D K^{1)}_C) \nabla_A K^{2)}_B - (\nabla_D K^{2)}_C) \nabla_A K^{1)}_) \tag{48}$$
$$= g^{AD}((\nabla_C K^{1)}_D) \nabla_A K^{2)}_B - (\nabla_C K^{2)}_D) \nabla_A K^{1)}_) \tag{49}$$
$$= 0 \tag{50}$$

Since g^{AD} is symmetric and we can relabel the repeated dummy index in the trace opertion in the second term in the third line

Similarly the other term is

$$\begin{aligned}
&g^{AD}((K^{1)}_D) \nabla_C \nabla_A K^{2)}_B - K^{2)}_D \nabla_C \nabla_A K^{1)}_B \\
&\quad + (K^{1)}_D) \nabla_B \nabla_A K^{2)}_C - K^{2)}_D) \nabla_B \nabla_A K^{1)}_C \\
&= g^{AD}((K^{1)}_D) \nabla_C \nabla_A K^{2)} - K^{2)}_D) \nabla_C \nabla_A K^{1)}_B \\
&\quad + (K^{1)}_D) \nabla_A \nabla_C K^{2)}_B - K^{2)}_D) \nabla_A \nabla_C K^{1)}_B \\
&+ K^{1)D} (R_B{}^X{}_{CA} + R_C{}^X{}_{BD}) K^{2)}_X - K^{2)D} (R_B{}^X{}_{CA} + R_C{}^X{}_{BD}) K^{1)}_X = 0
\end{aligned}$$

due to the symmetry of the curvature tensor and because $K^{1)}$, $K^{2)}$ are both Killing vectors.

(in the first line, writing this out in components we find that the Γ symbols cancel out and in the second line we use that $\nabla_A g_{BC} = 0$

If all the components of a metric are independent of some coordinate, then that coordinate axis tangent vector is a Killing vector. Eg, let us assume that

g_{ij} are all independent of x^1 the first coordinate. Then if we take the vector $V^i = (1,0,0...)$ we have

$$\pounds_V g_{ij} = V^k \partial_k g_{ij} + g_{ik} \partial_j V^k + g_{jk} \partial_i V^k = V^1 \frac{\partial g_{ij}}{\partial x^1} + 0 + 0 = 0 \qquad (51)$$

since all components of V^i are constants, and g_{ij} are independent of x^1.

Similarly, the Lie derivative of one coordinate axis tangent vector by another is zero, since in that coordinate system the components of each tangent vector are constants and thus have zero derivative.