Physics 530-11

Orbits in Schwartzschild Metric

Let us use the ordinary t, r coordinates for the Scwartzschild metric

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\Omega^2 \tag{1}$$

The Geodesic equations are given by

$$\frac{d}{d\tau}\left[(1 - \frac{2M}{r})\frac{dt}{d\tau}\right] = 0 \tag{2}$$

$$\frac{d}{d\tau}\left[r^2 \sin^2(\theta)\frac{d\phi}{d\tau}\right] = 0 \tag{3}$$

$$\frac{d}{d\tau}\left[r^2 \frac{d\theta}{d\tau}\right] - r^2 \sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{4}$$

$$(1 - \frac{2M}{r})\frac{dt}{d\tau}^2 - \left(\frac{dr}{d\tau}\right)^2 \frac{1}{(1 - \frac{2M}{r})} - r^2\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2\right) = [\pm 1, 0]$$

where the last equation is just the condition that τ is the proper time along the curve for either a timelike (+1), a spacelike (-1) or a null (0) geodesic. (Note that these equations are valid even if we take them so that the last equation is a constant on the RHS. The parameter τ would then not be the proper time or proper distance, but what is called an affine parameter along the curve.)

The first and second equation can be integrated to give

$$\frac{dt}{d\tau} = \frac{E}{1 - \frac{2M}{r}} \tag{5}$$

$$\frac{d\phi}{d\tau} = \frac{m}{r^2 \sin^2(\theta)} \tag{6}$$

where E and m are integration constants.

These are just two particular instances of the fact that, if ξ^A is a Killing vector, then

$$\frac{D}{D\tau}(\xi^A u^B g_{AB}) = (D_C \xi_B)u^C u^B + \xi_B \frac{Du^B}{D\tau} \tag{7}$$

The Killing equation is $D_C\xi_B + D_B\xi_C = 0$, which is the condition that the Lie derivative of the metrix along the vector field ξ is zero (which sets the first term to zero) . u^A, the tangent vector to a geodesic, is parallel transported along the geodesic, which sets the last term equal to zero. Thus $\xi^A u^B g_{AB}$ is a constant of the motion. Also, if we have a group of Killing vectors, $\{\xi_a^A\}$, where a labels the Killing vectors, then if $\kappa^{AB} = \sum_a \xi_a^A \xi_a^B$, $\kappa_{AB} u^A u^B$ will be a constant of the motion. This is a special case of a spacetime having a Killing Tensor, κ_{AB}, which is symmetric and obeys $D_{(A}\kappa_{BC)} = 0$. As above, this Killing tensor will define a (quadratic) constant of the motion for the spacetime. There exist spacetimes which have a Killing tensor not generated by a set of Killing vectors. (The spacetime of a rotating empty spce– the Kerr metric– is an example of a spacetime with a Killing tensor without it being generated by a Killing vector. It is this Killing tensor which allows the Kerr spacetime to have a complete set of integrals of the motion, allowing one to write again reduce the geodesic equations to quadrature.) (It has two killing vectors, time independence, and rotation in around the z axis (or ϕ independence), which give two constants of the motion, the Killing tenzor gives a third, and the length of the tangent vector to the geodesic gives a fourth constant of the motion. Since there are four equations, these four constants completely determine the geodesic in terms of first derivative equations for the four coordinates.

For the three rotational Killing vectors, κ^{AB} has components $\kappa^{\theta\theta} = 1$, $\kappa^{\phi\phi} = \frac{1}{sin(\theta)}^2$ and $\kappa^{\theta\phi} = 0$. Thus $\kappa_{AB} u^A u^B$ is constant along the geodesic., or, taking the constant to be l^2 we have,

$$l^2 = r^4 \left(\left(\frac{d\theta}{d\tau} \right)^2 + \sin(\theta)^2 \left(\frac{d\phi}{d\tau} \right)^2 \right) \tag{8}$$

is also constant of the motion. If we call ξ_ϕ^A, the Killing vector of translations in the ϕ component, such that $\xi^A df_A = \partial_\phi f$, and choose $\xi_A u^A = l$, the square root of the value of the rotational Killing tensor constant of the motion, we immediately get that $\left(\frac{d\theta}{d\tau}\right)^2 = l^2(1 - \frac{1}{\sin(\theta)^2})$ which is non-negative on the right only if $\theta = \pi/2$.

To return to the geodesic equations, define $u = \frac{1}{r}$ (do not confuse this u with the null coordinate used elsewhere or with the tangent vector to the geodesic– I choose this notation only because it is conventional in many papers and text books)) to give

$$\left(\frac{du}{d\phi}\right)^2 = -(1-2Mu)u^2 + \frac{E^2}{l^2} - \frac{[\pm 1, 0]}{l^2}(1-2Mu) \tag{9}$$

We can write this as

$$\left(\frac{du}{d\phi}\right)^2 = 2M(u-u_1)(u-u_2)(u-u_3) \tag{10}$$

where

$$2M(u_1+u_2+u_3) = 1 \tag{11}$$

$$2M(u_3(u_1+u_2)+u_1u_2) = [\pm 1, 0]\frac{2M}{l^2} \tag{12}$$

$$2M(u_1u_2u_3) = -\frac{E^2-[\pm 1, 0]}{l^2} \tag{13}$$

If u_1 and u_2 are very small,(the orbit is is large), then u_3 is close to $1/2M$. Note that, for the LHS of the equation for $(\frac{du}{d\phi})^2$ to be positive, and we label the roots such that $u1 \leq u2 \leq u3$, u must either lie between u_1 and u_2, or be larger than u_3. We will concentrate on the former case.

0.1 Perihelion advance

A. Einstein "Erklaerund def Perihelbewegung des Merkur aus der allgemeinen Relaivitaetstheory" Koeniglich PreussischeAkademie der Wissenshafen (Berlin) Sitzungsberichte (Nov 18 1915)

One of the first calculations Einstein did was to calculate the Perihelion advance of Mercury, becoming extremely excited when the answer from the theory agreed with the measured Perihelion advance minus the Newtonian effect from the gravitational perturbations by the rest of the planets (particularly Venus and Jupiter). He calculated it in a linearized approximation to his (earlier) theory in which R_{ij} was the key. He does not give any details as to how he derived the linearized solution he used. He was lucky in that, of the many possible solution to the linearized equations (differing by small coordinate transformations) he chose the one that worked, and which was the linearization of the solution found by Schwartzschild a few months later.

In the following we will assume the Schartzschild solution and carry out the calculation with that solution.

Let us look at the timelike case first, and choose E very close to 1 and l large. There will be one root, which I take to be u_3, which will lie very close to $\frac{1}{2M}$. The other two roots will be for very small values of u_1, u_2. Define $\bar{u} = (u_1 + u_2)/2$ and $\Delta = (u_2 - u_1)/2$ and I will assume that both $\bar{u}$ and Δ are positive and very small. (For Mercury, $2M\bar{u} \approx 5 \cdot 10^-8$) If $\bar{u} = \Delta$ this corresponds to a parabolic orbit in the newtonian case, while if $\bar{u} < \Delta$ the Newtonian orbit is hyperbolic.

Then $u_3 = \frac{1}{2M} - 2\bar{u}$. We get

$$(\frac{du}{d\phi})^2 = -(1 - 6M\bar{u})((u - \bar{u})^2 - \Delta^2) + 2M(u - \bar{u})((u - \bar{u})^2 - \Delta^2) \quad (14)$$

For very small $\Delta << \bar{u}$, and $2M\bar{u} << 1$, the last term is small, and we have

$$u \approx \bar{u} + \Delta \cos(\sqrt{1 - 6M\bar{u}}\phi + M\Delta sin(\sqrt{1 - 6M\bar{u}}\phi)) \quad (15)$$

Neglecting the periodic term in the argument, this is an elliptical orbit with a perihelion precession of $2\pi(3M\bar{u})$ per orbit. This was the first "test". Einstein had his new theory more or less completed, when he calculated this. (He used the first order linearized theory to do since Schwartzschild and Droeste had not yet come up with the exact solution). He was highly excited when he discovered that it fit the experimental results. His previous Entwurf theory had produced a figure of only 18 seconds of arc per century.

For Mercury the perihelion advance is about 42 sec of arc per century which is, within about 5% what Einstein's theory gave. Using sattelite orbits with their transponders, one can now reduce this down to much less than 1%. For

Note that one must be careful. In this case the linear terms in u and the constant give

$$\frac{2M}{l^2} = 2Mu_3(u_1 + u_2) + u_1u_2 = (1 - 4M\bar{u})2\bar{u} + 2M(\bar{u}^2 - \Delta^2) \quad (16)$$

$$-(E^2 - 1)/l^2 = 2Mu3(u_1u_2) = (1 - 4M\bar{u}_(\bar{u}^2 - \Delta^2) \quad (17)$$

Note that by an approprite choise of E^2 (which must be less than 1 for a finite orbit which does not go to $u = 0$ or $r = \infty$) and l^2 we can create a near circular obit ($\bar{u} > 0$ and $\Delta^2 << \bar{u}^2$.

For a while, in the 60s and 70s, there was a claim by Dicke that the sun was not spherical but oblate– due presumably to a rapidly rotating core. That oblateness would have produced a gravitational field with a $1/r^3$ Newtonian potential which would have caused some perihelion advance even in Newtonian gravity. He used this to justify his "Brans-Dicke" theory, a sort of amalgum of the Nordstrom and Einstein theories (ie, it had a conformal factor to the metric as well as the Einstein type metric which led to a reduced perihelion advance, and a reduced light deflection). The justification diappeared when the solar oscillations were recognized, and the oblateness of the sun disappeared. (Henry Hill measured the oblateness and found a much smaller value than Dicke did. In the course of his measurements he realised that the sun had oscillatory modes, and Dicke must have measured the sun when the quadrapolar oscillations were at their maximum. Since these oscillation are much faster than the orbital period of even mercury, they will average out in their effect on the orbit.)

0.2 Light deflection

For massless particles, we get

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{l^2} - (1-2Mu)u^2 = 2M(u-u_3)(u-u_1)(u-u_2) \tag{18}$$

where again

$$u_3 = \frac{1}{2M} - 2\bar{u} \tag{19}$$

but the absense of the term in the potential proportional to u gives us

$$u_3(2\bar{u}) + \bar{u}^2 - \Delta^2 = 0 \tag{20}$$

or

$$\bar{u} = M(3\bar{u}^2 + \Delta^2) \tag{21}$$

Ie, if $M\Delta$ is small, then $\bar{u}$ is very small, (of order $M\Delta^2$). We can write

$$\left(\frac{du}{d\phi}\right)^2 = (1-6M\bar{u})(\Delta^2 - (u-\bar{u})^2) - 2M(u-\bar{u})(\Delta^2 - (u-\bar{u})^2) \tag{22}$$

But since $\bar{u} \approx M\Delta^2$ to lowest order in $M\Delta$, the term $M\bar{u}$ is of order in $(M\Delta)^2$ and can be neglected. This gives

$$\left(\frac{du}{d\phi}\right)^2 = (\Delta^2 - (u-\bar{u})^2 - 2M(u-\bar{u})(\Delta^2 - (u-\bar{u})^2) \tag{23}$$

or,

$$\int \frac{du}{\sqrt{\Delta^2 - (u-\bar{u})^2}} = \sqrt{1 - 2M(u-\bar{u})}d\phi \tag{24}$$

Integrating over u from 0 to $\Delta + \bar{u}$ to 0, recalling that the total deflection is that coming in toward the star and then going back out (or twice that of going out), and using that $u - \bar{u} \approx \Delta\cos(\phi)$, we get

$$2\arccos(-\frac{\bar{u}}{\Delta}) = \phi - 2M\Delta \tag{25}$$

or

$$\phi = (\pi + 2\frac{\bar{u}}{\Delta}) + 2M\Delta = \pi + 4M\Delta \tag{26}$$

Ie, in passing by the sun, the total angle is greater, by $4M\Delta$ than that for a straight line. This is the light deflection.

This has been accurately measured to better than 0.1% by the Hipparchus sattelite, which can see the light deflection ($\approx 1milliarcsec$) of stars at 90 degrees from the sun. (It measured (1989-1994) the relative angle between a large number (¿100,000) of bright stars to about $1m$arc sec, and one must use the deflection of light by the sun to fit the positions of the stars. A future sattelite (2012), Gaia, will measure down to 7μas and should measure the light deflection to the 10^{-6} level by which point higher order effects become important as well.

0.3 Shapiro Time delay

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See Irwin I. Shapiro (1964). "Fourth Test of General Relativity". Physical Review Letters. 13 789

Irwin I. Shapiro et al (1968). "Fourth Test of General Relativity: Preliminary Results". Physical Review Letters. 20 1265

Consider the equation for the time delay along the path

$$\frac{dt}{d\tau} = \frac{E}{1-\frac{2M}{r}} \tag{27}$$

or

$$\frac{dt}{d\phi} = \frac{E}{l}\frac{1}{u^2(1-2Mu)} \approx \frac{E}{l}\left(\frac{1}{u^2}+\frac{2M}{u}\right) \tag{28}$$

There are three terms here. E/l is a function of M when written in terms of $\bar{u}$ and Δ. In particular $-2M(u_3)(u_1)(u_2) = \frac{E^2}{l^2}$, or

$$(1-4M\bar{u})(\bar{u}^2-\Delta^2) = -\frac{E^2}{l^2} \tag{29}$$

which, since $\bar{u} = 2M\Delta^2$ to lowest order in M, we have $\Delta^2 + O((M\Delta)^2) = \frac{E^2}{l^2}$.

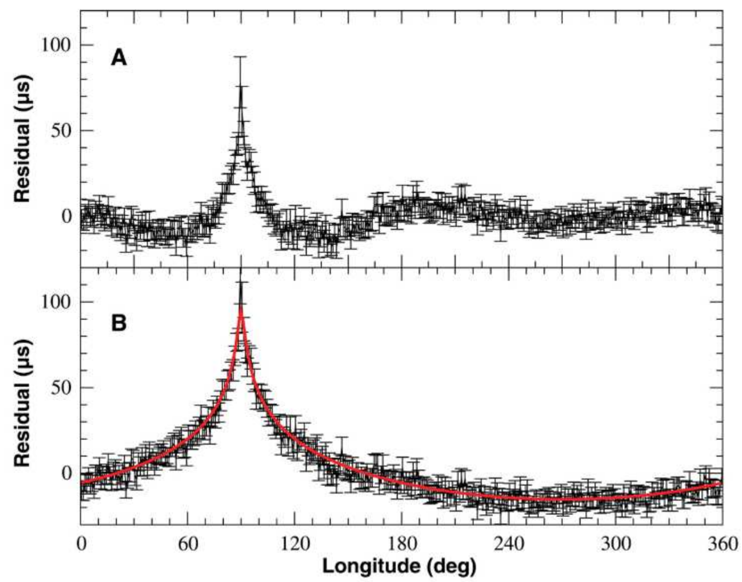
Since the straight line, $u = \Delta\cos(\phi)$ would give the time along the shortest path in the flat metric, any alteration of that path will alter the time only to second order in that alteration of the path. Ie, any deflection from that straight line will increase the length of that path, and thus the flat space time only to second order in that deflection. Thus, taking the straight line in u, ϕ space will give the value for that first term with corrections only to second order in the deflection (and thus only to second order in M). The second term is the correction due to the fact that the time goes more slowly nearer the black hole (although the spatial part of the metric does play a role in the time delay as well). Thus we get

$$\begin{aligned}
\delta t &= \int \frac{1}{\Delta\cos(\phi)^2}d\phi + 2M\int\frac{d\phi}{cos(\phi)} && (30)\\
&= \frac{1}{\Delta}\tan(\phi) + 2M\ln\left(\frac{1+\sin(\phi)}{\cos(\phi)}\right) && (31)\\
&= \frac{1}{\Delta}(\tan(\phi_2)-\tan(\phi_1)) + 2M\ln\left(\frac{(1+\sin(\phi_1))\cos(\phi_2)}{(1+\sin(\phi2))\cos(\phi_1)}\right) && (32)
\end{aligned}$$

Note that this is the change in travel time in t coordinates, and does not take into account the change in proper time at either the source or the receiver.

If we have a clock orbiting a star, as the orbiting star runs so that the impact parameter ($\frac{1}{\Delta}$) becomes small, the angles ϕ_1 and ϕ_2 go very close

Fig. 2. Measurement of a Shapiro delay demonstrating the curvature of space-time.



M Kramer et al. Science 2006;314:97-102

to $-\pi/2$ and $\pi/2$ respectively, and the logarithm becomes large. This logarithmic dependence of the time delay on the motion of the background star is what one fits. One of the most wonderful demonstrations of this is the double binary pulsar PSR J0737-3039A/B which has an orbit almost edge on. The rotation of the one pulsar (A) is the clock which beats extremely regular time. Because the orbit is almost edge on the light passes very close to B, suffering the time delay. The top plot is the best fit of the data to the orbit if one does not take the shapiro time delay into account– a lousy fit. The lower plot is the residuals (data-minus theory) for the best fit including the Shapiro time delay, but with the shapiro time delay added back into the residuals to show what its size is. The red line is the best fit, and the points are the residuals. (Tests of General Relativity from Timing the Double Pulsar M. Kramer, et al. Science 314, 97 (2006))

Notes on derivation:

While I cannot see the problem in that above derivation of the Shapiro effect, I do not entirely trust it. So let us look at a more direct (but messy) derivation.

The radial equation for light rays is

$$\left(\frac{du}{d\phi}\right)^2 = (1-6M\bar{u})(\Delta^2-(u-\bar{u})^2) - 2M(u-\bar{u})(\Delta^2-(u-\bar{u})^2) \quad (33)$$

which to first order in M gives ($6M\bar{u}$ is of order $(M\Delta)^2$ and to zeroth order in M, $u-\bar{u}$ is $\Delta\cos(\phi)$)

$$\arccos\left(\frac{u-\bar{u}}{\Delta}\right) = \phi - M\Delta\sin(\phi) \quad (34)$$

or

$$u = \bar{u} + \Delta\cos(\phi - M\Delta\sin(\phi)) \approx \bar{u} + \Delta\cos(\phi) + M\Delta^2 sin(\phi)^2 \quad (35)$$

I have chosen $\phi = 0$ to be the angle when the particle is closest to the center (u is a maximum).

The time delay from $\phi = 0$ (the closest approach to the star) to the location of the transmitter/receiver is then given by (recalling that $E/l = \Delta$)

$$\begin{aligned}
\Delta t &= \int_0^{\phi_i} \frac{\Delta d\phi}{\left(\bar{u}+\Delta\cos(\phi)+M\Delta^2\sin(\phi)^2\right)^2(1-2M\Delta\cos(\phi))} && (36)\\
&\approx \int \frac{1}{\Delta\cos(\phi)^2} - 2M\frac{(1+\sin(\phi)^2)}{\Delta\cos(\phi)^3} + \frac{2M}{\cos(\phi)}d\phi && (37)\\
&= \frac{1}{\Delta}\tan(\phi_i) - 2M\left(\frac{\sin(\phi_i)}{\cos(\phi)^2}\right) + 2M\ln\left(\frac{1+\sin(\phi_i)}{\cos(\phi_i)}\right) && (38)
\end{aligned}$$

(This expansion is only valid if $cos(\phi) >> M\Delta$).

Now, Δ is determined by the final location of the transmitter/receiver. Ie, we have

$$u_i = \bar{u} + \Delta\cos(\phi_i) + M\Delta^2\sin(\phi_i)^2 \quad (39)$$

or

$$\Delta \approx \frac{u_i}{\cos(\phi_i)} - Mu_i^2\frac{1+\sin(\phi_i)^2}{\cos(\phi_i)^3} \quad (40)$$

which gives

$$\Delta t_i \quad = \quad \frac{\sin(\phi_i)}{u_i} - M\sin(\phi_i) + 2M\ln\left(\frac{1+\sin(\phi_i)}{\cos(\phi_i)}\right) \tag{41}$$

Note that this differs from the previous expression by the $-M\sin(\phi_i)$ term which is essentially a constant since for $u_i << \Delta$ (the emission or observation point much further away that the point of closest approach), $\sin(\phi_i) \approx 1$.

If the light starts off at u_1, ϕ_1 and ends at u_2, ϕ_2 we need that Δ be the same for both

$$\frac{u_1}{\cos(\phi_i)}\left(1 + Mu_1\frac{1+\sin(\phi_1)^2}{\cos(\phi_1)^2}\right) = \frac{u_2}{\cos(\phi_i)}\left(1 + Mu_2\frac{1+\sin(\phi_2)^2}{\cos(\phi_2)^2}\right) \tag{42}$$

The total angle $\phi_1 + \phi_2$ must be the total angle between the receiver and the emitter. To zeroth order in M, we have

$$\frac{u_1}{\cos(\Phi_1)} = \frac{u_2}{\cos(\Phi_2)} = \bar{\Delta} \tag{43}$$

To next order, we take $\phi_1 = \Phi_1 + \delta\phi$, $\phi_2 = \Phi_2 - \delta\phi$ and can find $\delta\phi$ which will be a term to first order in M.

The total time is now

$$\Delta t \quad = \quad \frac{\sin(\Phi_1 + \delta\phi)}{u1} + \frac{\sin(\Phi_2 - \delta\phi)}{u2} - M(\sin(\Phi_1) + \sin(\Phi_2)) \tag{44}$$

$$+ 2M\ln\left(\frac{(1+\sin(\Phi 1))(1+\sin(\Phi_2))}{\cos(\Phi_1)\cos(\Phi_2)}\right) \tag{45}$$

To lowest order in $\delta\phi$ (which is proportional to M) we have $(\frac{\cos(\Phi_1}{u_1} - \frac{\cos(\Phi_2)}{u_2})\delta\phi = 0$ so the change in the angle at which the light achieves its angle of closest approach makes no difference to the time.

This can also be written as

$$\Delta t = \frac{\sin(\Phi_1)}{u_1 + Mu_1^2} + \frac{\sin(\Phi_2)}{u_2 + Mu_2^2} + 2M\ln\left(\frac{(1+\sin(\Phi 1))(1+\sin(\Phi_2))}{\cos(\Phi_1)\cos(\Phi_2)}\right) \tag{46}$$

This is not the same as the expression I wrote down before. It differs by the terms $-M(sin(\phi_1) + \sin(\phi_2))$ which is of order M^2 if orbit is far from the $r = 2M$

0.4 Large Mass equivalence principle (Nordtvedt Effect)

One of the questions that arises as far as the equivalence principle is concerned is the ratio of the gravitational to inertial mass for large bodies. It can be proven that small bodies follow geodesics (if we neglect the self gravitational field of a body, the conservation of the stress energy tensor implies that the body follow a geodesic, if there is no higher multipole moment interaction with the curvature. However, if the self gravitational field becomes significant (usually phrased that the gravitational potential energy becomes a significant part of the energy of the body), perhaps the gravitational to inertial mass ratio is no longer 1.

This has been tested on the moon-earth system. Consider the moon, mass m, and the earth mass M orbiting the sun. The Lagrangian can be written , with α_m and α_M the gravitational to inertial mass ratio of the moon and earth, $\mathcal{M}$ the mass of the sun, $\vec{R}$ the distance from the center of mass of the earth moon system to the sun, and $\vec{x}$ the separation of the earth moon system.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}((m+M)\dot{\vec{R}}^2 + \bar{M}\dot{\vec{x}}^2) + \alpha_m\alpha_M\frac{GMm}{x} \\ &\quad +G\mathcal{M}\left(\frac{\alpha_m m}{|\vec{R}+\frac{M}{M+m)}\vec{x}|} + \frac{\alpha_M M}{|\vec{R}-\frac{m}{M+m}\vec{x}|}\right) \\ &\approx \frac{1}{2}((m+M)\dot{\vec{R}}^2 + \bar{M}\dot{\vec{x}}^2) + +\alpha_2\alpha_2\frac{GMm}{|\vec{x}+\beta\vec{R}|} + G\mathcal{M}\left(\frac{\alpha_m m+\alpha_M M}{|\vec{R}|}\right)\end{aligned} \tag{47}$$

where

$$\beta = (\alpha_m - \alpha_M)\left(\frac{|\vec{x}|}{|\vec{R}|}\right)^3\frac{\mathcal{M}}{M+m} \tag{48}$$

(to lowest order in $\frac{|\vec{x}|}{|\vec{R}|}$ and $(\alpha_m - \alpha_M)$). We can choose the solution such that $|\vec{R}|$ is constant, and $|\vec{x}|$ is almost constant– ie the lowest order solution– and such that the direction of $\vec{R}$ rotates at a rate of once a year (ie much more slowly than the rotation rate of the moon about the earth). The change of the direction of R will then be adiabatic. The solution will thus be that the moon rotates not about the center of mass of the earth moon system, but at a displaced point with the displacement pointing toward the sun.

This polarization of the earth moon system has been measured with the set of corner reflectors left on the moon by the Apollo 11, 14 and 15 missions and the Lunokhod 1 and 2 Soviet Rover missions. With long term averaging the distance from the earth to the moon with period of 1 year can be measured to less than 1mm when averaged over a months accuracy and the results are consistant with zero polarization. This gives $\alpha_m - \alpha_M$ less than about 10^{-14} which is much smaller than the difference in the ratio of the gravitational self energy over the total mass of the earth and moon. Ie, gravitational energy gravites in the same way that ordinary energy gravitates. (See the online talk slides by J Müller http://cddis.gsfc.nasa.gov/lw16/docs/presentations/sci_8_Mueller.pdf)

The Brans Dicke theory would predict that $\alpha_m - \alpha_M$ would be different from zero, and this experiment has been the key one to rule out that theory.