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Let us embed a Spherically symmetric empty solution (Schwartzschild) into a flat (k=0) dust cosmology. The dust is used since any of the others have an internal pressure which, on the surface of joining between the matter filled cosmology and the empty space would result in infinite accelerations due to the pressure step. The k=0 is used because it is the easiest to calculate (the others can be used but they are more complicated).

The interior Schwarzschild metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{1}$$

while the cosmological metric is

$$ds^2 = -dT^2 + T^{4/3}(dR^2 + R^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{2}$$

The angular coordinates can be left alone. The junction between the two will occur at $R = R_0$, which is a geodesic of the cosmology. Note that the "radius" of the spherical symmetry has a circumpherence $/2\pi$ of $T^{2/3}R_0$, and where T is the proper time along that geodesic.

Looking at the radial geodesics in the Schwarzschild metric, we have

$$\frac{dt}{ds} = \frac{E}{1 - \frac{2M}{r}} \tag{3}$$

$$\frac{(\frac{dr}{ds})^2 = E^2 - 1 + 2M}{r} \tag{4}$$

where s is the proper time along that geodesic and r is the circumpherence coordinate. Thus we want the geodesic in Schwartzschild to have s depencence of $r = R_0 s^{2/3}$. Plugging into the second equation we have

$$\frac{4}{9}R_0^2 s^{-2/3} = (E^2 - 1) + \frac{2M}{R_0 s^{2/3}} \tag{5}$$

from which we find that $E^2 = 1$ and $\frac{4}{9}R_0^3 = 2M$

Let us change coordinates. We have

$$\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = \frac{dr}{dt}\frac{1}{1 - \frac{2M}{r}} \tag{6}$$

or

$$\frac{dr}{dt} = \sqrt{\frac{2M}{r}}\frac{r - 2M}{r} \tag{7}$$

is the equation for the geodesics in Schwartzschild spacetime. Define a new coordinate

$$R - R_0 = t - \int \frac{dr\sqrt{r^3}}{\sqrt{2M}(r - 2M)} \tag{8}$$

The coordinate R is constant along the geodesics (just as the coordinate R is constant along the geodesics in the cosmological spacetime). Choose the geodesic which defines the boundary between the empty space and the cosmology as $R = R_0$.

The Schwartzschild metric becomes

$$-(1-\frac{2M}{r})(dR+\frac{dr}{\sqrt{\frac{r^3}{2M}}(r-2M)})^2+\frac{dr^2}{1-\frac{2M}{r}}-r^2(d\Omega^2) \tag{9}$$

$$=-(1-\frac{2M}{r})dR^2-2\frac{dRdr}{\sqrt{\frac{r}{2M}}}-\frac{rdr^2}{2M}+r^2(d\Omega^2) \tag{10}$$

Now, define

$$T=\frac{2}{3}\sqrt{\frac{r^3}{2M}}+R-R_0 \tag{11}$$

The Schwartzschild geometry becomes

$$ds^2=-dT^2+\frac{2M}{r}dR^2+r^2d\Omega^2 \tag{12}$$

where

$$r=\left[\frac{3\sqrt{2M}}{2}(T-R+R_0)\right]^{2/3} \tag{13}$$

The requirement that r be a continuos function of R across R_0 gives

$$M=\frac{2}{9}R_0^3 \tag{14}$$

Given this choice, the $g_{TT},\ g_{\theta\theta},\ g_{\phi\phi}$ compnents are continuous across the surface $R=R_0$. The g_{RR} are not, but this is not required. A more detailed calculation shows that this metric and the curvature are well behaved at $R=R_0$ (no delta functions), despite the apparent discontinuity of some of the metric coefficients, but a proof of this would lead us too far afield.

Note that there is no evidence whatsoever in the Schartzschild protion of the metric of anything occuring at r=2M. The metric is completely regular there, as one would expect.

One could carry out an almost identical analysis for the dust cosmology being inside ($R<R_0$ while the exterior was Schwartzschild. This would then be a model of a star emerging out of the past horizon of a black hole (a white hole) or of a star collapsing, if one took the contracting rather than expanding universe as the model.

The above procedure canalso be used for the $k=\pm1$ cosmologies, and fit them onto a Schwartzschild metric.

Since outside the surface, the cosmology is identical to what it would be without the black hole, and since the cosmology is homogeneous, one could imagine scooping out many such voids in the dust cosmolgy. Those voids could then describe the empty space around say galaxies or stars, and create in this way a model of cosmology which was homogeneous and isotropic in teh large, but contained within it anisotropies and inhomogeneities (the Schwartzschild voids). This is called the Swiss Cheese cosmology, and as an exact solution of Einstein's equations, can be used to test out various ideas about the effect of the small scale clumpiness of the universe on our observations of the cosmology.

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