Physics 407-07
Vectors

1) Tangent and cotengent (gradient) vectors.

Lets say we have a 2-D spacetime, with coordinates t, x. Ie, $t(p)$ and $x(p)$ are nice functions, with the equations $t(p) = t_0, \quad x(p) = x_0$ having a unique solution p_0 and such that if if t_1, x_1 is different from t_0, x_0 then the solution p_1 is not the same as p_0. The consider the the curves

$$\{t(\gamma(\lambda)), x(\gamma(\lambda))\} = \{\lambda, \lambda\} \tag{1}$$

$$\{t(\tilde\gamma(\lambda)), x(\tilde\gamma(\lambda))\} = \{\sin(\lambda), \frac{1}{2} sign(\lambda)\sqrt{2\cos(\lambda) - 2)}\} \tag{2}$$

$$\{t(\tilde{\tilde\gamma}(\lambda)), x(\tilde{\tilde\gamma}(\lambda))\} = \{\sqrt{\lambda^2}, 0\} \tag{3}$$

Then $f(p(t,x)) = \hat f(t,x)$ and $f(\gamma(\lambda)) = \hat f(t(\gamma(\lambda), x(\gamma(\lambda))$ and $df\partial_\gamma = \frac{d}{d\lambda}\hat f(t(\lambda), x(\lambda)) = \frac{dt(\lambda)}{d\lambda}\partial_t f(t(\lambda), x(\lambda)) + \frac{dx(\lambda)}{d\lambda}\partial_x f(t(\lambda), x(\lambda))$ Plugging in for $\gamma, \tilde\gamma, \tilde{\tilde\gamma}$ from above, we find that the tangent vectors for γ and $\tilde\gamma$ are the same at $\lambda = 0$ (at $\lambda \neq 0$ the curves do not go through the same point, so we cannot compare their tangent vectors). However the tangent vectors to γ and $\tilde{\tilde\gamma}$ and γ are not the same. at the point p_0.

Equality or otherwise of two tangent vectors is defined only of the two curves go through the same point, and only at that point where the two curves go through the same point.

Note, that if we add a constant to the parameter λ, this does not change the tangent vector at any point that the curve goes through, although the value of λ at a point that the curve goes through will change if we add a constant to lambda. Ie if $\tilde\lambda = \lambda + c$, then the tangent vector at $\lambda = \lambda_0$ for $\gamma(\lambda)$ and $\tilde\gamma(\tilde\lambda) = \gamma(\lambda + c)$ at the point $p_0 = \gamma(\lambda_0) = \tilde\gamma(\lambda_0 + c)$ will be the same. .

Geometric representation:

WE can represent a tangent vector at the p_0 by taking one of the curves which has that tangent vector at the point $p_0 = \gamma(\lambda_0)$ and draw the curve from λ_0 to $\lambda_0 + 1$. Then at the $\lambda_0 + 1$end of that curve draw a little arrow head, to indicate in which direction along the curve the parameter increases. It does not matter which of the curves one chooses, because at this point there is nothing which can choose one curve over the other.

The contangent vector can be represented by taking one of the functions f(p) and drawing a little surface– ie all of the points in the vicinity of p_0 such vector-drwaings.png

that $f(p) = f(p_0)$ one this surface also put a dot on the point p_0. Then draw another surface which is the points p such that $f(p) = f(p_0) + 1$.

Metric:

Consider the the curve $\gamma(\lambda) = p(x^i(\lambda))$. The tnagent vector $\partial_{\gamma(\lambda)} = \sum_i(\frac{dx^i(\lambda)}{d\lambda}\partial_{x^i})$. and $g_{AB}\partial^A_\gamma\partial^B_\gamma = \sum_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}g_{AB}\partial^A_{x^i}\partial^B_{x_j}$ Now, we define the real numbers $g_{ij} = g_{AB}\partial^A_{x^i}\partial^B_{x_j}$ as the components g_{ij} which gives us

$$g_{AB}\partial^A_\gamma\partial^B_\gamma = \sum_{ij} g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \tag{4}$$

Note that the left and right side look almost identical, except that on the left there is not summation over A, B and and on the right there are numbers rather than abstract vectors and tensors. If we interpret $\frac{dx^i}{d\lambda}$ as another symbol for ∂_A, we find that the and the repeated symbols A and B as also standing for summation over those as interpreted as indices, we have essentially the same notation on both sides. Ie, since the two sides are equal, we can feel free to interpret either side as numbers or function, or as geometric vectors and tensors. This is Penrose's abtract index notation.

Geodesics:

The variation of the length of the curve equal to zero is the geodesic condition.

$$\delta L = \delta\int_{\lambda_0}^{\lambda_1}\sqrt{g_{AB}\partial^A_\gamma\partial^B_\gamma}d\lambda \tag{5}$$

where the variation is allowing the curve $\gamma(\lambda)$ to depend on another parameter, say ϵ, such that $\gamma(\epsilon = 0, \lambda) = \gamma(\lambda)$ the curve we want to find, and $\gamma(\epsilon, \lambda_0)$ and $\gamma(\epsilon, \lambda_1)$ are both independent of ϵ. Then the variation condition is that

$$\delta L = \frac{dL}{d\epsilon}|_{\epsilon=0} \tag{6}$$

Using the expression for $g_{AB}\partial^A_\gamma\partial^B_\gamma$ as a function of the coordinates, and the curve γ as a function of the coordinates as $dx^i\epsilon, \lambda$ we get

$$\frac{dL}{d\epsilon} = \int_{\lambda_0}^{\lambda_1}\frac{1}{2S}\sum_{ij}\left(\sum_k\frac{dx^k}{d\epsilon}(\partial_k g_{ij})\frac{dx^i}{d\lambda}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right. \tag{7}$$

$$+g_{ij}\frac{d^2x^i}{d\epsilon d\lambda}\frac{dx^j}{d\lambda}+g_{ij}\frac{dx^i}{d\lambda}\frac{d^2x^j}{d\epsilon d\lambda}\Bigg)d\lambda \tag{8}$$

Doing an integration by parts to get rid of the derivates with respect to both λ and ϵ, and using that a sum over an index is independent of name of the index, we can write this as

$$\frac{dL}{d\epsilon}=\sum_k\int_{\lambda_0}^{\lambda_2}\int_{\lambda_0}^{\lambda_i}\frac{dx^k}{d\lambda}\Bigg(\sum_{ij}\frac{1}{S}g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \tag{9}$$

$$-\frac{d}{d\lambda}\left[\sum_j(\frac{1}{S}g_{kj}\frac{dx^j}{d\lambda})+\sum_i(\frac{1}{S}g_{ik}\frac{dx^i}{d\lambda})\right]\Bigg)d\lambda \tag{10}$$

$$+\frac{dx^k(\epsilon,\lambda_1)}{d\epsilon}...-\frac{dx^k(\epsilon,\lambda_0)}{d\epsilon} \tag{11}$$

where $S=\sqrt{g_{AB}\partial^A_\gamma\partial^B_\gamma}$ the length of the tangent vector. The last line (from the boundary terms in the integration by parts) is zero. The terms muliplying $\frac{dx^k}{d\epsilon}$ in the first parts must be zero individually for each epsilon, because the functionality with respect to ϵ is completely arbitrary

This the geodesic equation must be

$$0 = \Bigg(\sum_{ij}\frac{1}{S}g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \tag{12}$$

$$-\frac{d}{d\lambda}\left[\sum_j(\frac{1}{S}g_{kj}\frac{dx^j}{d\lambda})+\sum_i(\frac{1}{S}g_{ik}\frac{dx^i}{d\lambda})\right]\Bigg) \tag{13}$$

If we define $s=\int Sd\lambda$, then this equation becomes

$$0 = \Bigg(\sum_{ij}g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \tag{14}$$

$$-\frac{d}{ds}\left[\sum_j(g_{kj}\frac{dx^j}{ds})+\sum_i(g_{ik}\frac{dx^i}{ds})\right]\Bigg) \tag{15}$$

Example:

Let us assume that the metric is

$$S^2 = r^2 \left(\frac{d\phi}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 \tag{16}$$

$$L^2 = \int_{\lambda_0}^{\lambda_2} S(\epsilon, \lambda) d\lambda \tag{17}$$

Taking the derivative with respect to ϵ and setting the multipliers in the integral multiplying $\frac{dr}{d\epsilon}$ and $\frac{d\phi}{d\epsilon}$ eqaul to 0 we get

$$0 = r(\frac{d\phi}{ds})^2 - \frac{d}{ds}(\frac{dr}{ds}) \tag{18}$$

$$0 = \frac{d}{ds}(r^2 \frac{d\phi}{ds}) \tag{19}$$

The second equation says that $r^2 \frac{d\phi}{ds}$ is constant, $(ds = Sd\lambda)$, which I will desigmate by ℓ. This makes the first equation

$$\frac{d^2 r}{ds^2} = \frac{\ell^2}{r^3} \tag{20}$$

Multiplying by $\frac{dr}{ds}$ we get $\frac{d}{ds}(\frac{1}{2}(\frac{dr}{ds})^2 + \frac{\ell^2}{r^2})) = 0$ But $Sd\lambda = ds$and thus

$$(\frac{dr}{ds}^2 + \frac{\ell^2}{r^2}) = 1 \tag{21}$$

Rewriting in terms of $\frac{dr}{d\phi}$ we get

$$(\frac{dr}{d\phi})^2 (\frac{d\phi}{ds^2})^2 + \frac{\ell^2}{r^2} = 1 \tag{22}$$

$$(\frac{du}{d\phi})^2 + u^2 = \frac{1}{l^2} \tag{23}$$

$$u = \frac{1}{\ell} \cos(\phi - \phi_0) \tag{24}$$

where $u = 1/r$. Thus, ℓ is the radius of closest aproach to r=0. Also,

$$\frac{d\phi}{ds} = \ell u^2 = \frac{1}{\ell} \cos^2(\phi - \phi_0) \tag{25}$$

$$\tan(\phi - \phi_0) = \frac{s - s_0}{\ell} \tag{26}$$

Note that if one were to take new coordinates, $x = r\cos(\phi), y = r\sin(\phi)$, the metric would have the form $S^2 = (\frac{dx}{d\lambda})^2 + (\frac{dy}{d\lambda})^2$, which is just the metric of flat 2-D space.

0.1 Components

Given our coordinates, and out definitions of vectors, we can express the vectors in terms of each other.

Consider the function f(p) defined near the point p_0. Let us define another function F(p) by

$$F(p) = f(p_0) + \sum_i \frac{\partial f(P(\{x^i\}))}{\partial x^i}|_{\{x^i = x^i(p_0)\}}(x^i(p) - x^i(p_0)) \tag{27}$$

It is possible to show that F(p) has the same cotangent vector as $f(p)$ has at the point p_0. I.e., for all curves $\gamma(\lambda)$, the derivative along the curve of these two functions is the same at the point p_0. This means that

$$df_A = dF_A \tag{28}$$

But

$$dF_A = \sum_i \frac{\partial f(P(\{x^i\}))}{\partial x^i}|_{\{x^i = x^i(p_0)\}} dx^i_A \tag{29}$$

since it is a sum with constant coefficients of the coordinate functions $x^i(p)$. Thus we can write

$$df_A = \sum_i \frac{\partial f(P(\{x^i\}))}{\partial x^i}|_{\{x^i = x^i(p_0)\}} dx^i_A \tag{30}$$

The coefficients, $\frac{\partial f(P(\{x^i\}))}{\partial x^i}|_{\{x^i = x^i(p_0)\}}$ are called the components of df_A in the coordinate system $\{x^i\}$.

Similarly we can for any curve γ write

$$(\frac{\partial}{\partial \gamma})^A = \sum_i \frac{dx^i(\gamma(\lambda))}{d\lambda} \left(\frac{\partial}{\partial x^i}\right)^A \tag{31}$$

Then the $\frac{dx^i(\gamma(\lambda))}{d\lambda}$ are the components of $(\frac{\partial}{\partial\gamma})^A$ in the coordinate system $\{x^i\}$.

Finally, we can see that

$$(\frac{\partial}{\partial x^i})^A dx^j_A = \delta^j_i \tag{32}$$

and thus

$$V^A W_A = \sum_i V^i W_i \tag{33}$$

This also shows that this product of the sums of components is independent of which coordinate system one happens to have chosen, because the left had side was defined without any reference to coordinates.

0.2 Metric

While the above structures are useful, in almost all of physics, another structure plays a crucial role, namely a metric. This is something which determines the size of things. The metric is defined as the generalisation of the dot product of two tangent vectors. In particular, given two tangent vectors V^A and W^B (again the value of the superscript does not matter). We thus define a function g of the two vectors

$$g(V^A, W^B) \tag{34}$$

to the real numbers as the "dot product" of two tangent vectors. We demand, primarily be analogy with the dot product, that this metric be linear in both arguments.

$$g(V^A, W^B + Z^B) = g(V^A, W^B) + g(V^A, Z^B) \tag{35}$$

and that g be symmetric

$$g(V^A, W^B) = g(W^B, V^A) \tag{36}$$

Now we define the length squared of a vector to be given by $g(V^A, V^A)$. This allows us also to define the dot product in terms of lenths

$$g(V^A, W^B) = \frac{1}{2}\left(g(V^A + W^A, V^B + W^B) - g(V^A, V^B) - g(W^A, W^B)\right) \tag{37}$$

Writing V^A and W^B in terms of coordinates components, we get

$$g(V^A, W^B) = \sum_{ij} V^i W^j g\left(\left(\frac{\partial}{\partial x^i}\right)^A, \left(\frac{\partial}{\partial x^j}\right)^B \right) \equiv \sum_{ij} V^i W^j g_{ij} \tag{38}$$

The numbers $g_{ij} = g(\left(\frac{\partial}{\partial x^i}\right)^A, \left(\frac{\partial}{\partial x^j}\right)^B)$ are called the components of the metric in the coordinate system x^j. Note again that the metric was defined independent of coordinates, and thus since the left had side is independent of coordinates, so must the sum of the right hand side be, even though the values of the coefficients clearly do depend on the coordinates.

0.3 Length of a curve

Given a curve $\gamma(\lambda)$, the lenght of the curve from the point $p_1 = \gamma(\lambda_1)$ to $p_2 = \gamma(\lambda_2)$ is defined to be

$$\int_{\lambda_1}^{\lambda_2} \sqrt{g\left(\left(\frac{\partial}{\partial \gamma}\right)^A, \left(\frac{\partial}{\partial \gamma}\right)^B \right)} d\lambda \tag{39}$$

$$= \int_{\lambda_1}^{\lambda_2} \sqrt{\sum_{ij} g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda \tag{40}$$

Note that because of the square root, the right side of this equation is independent of the parameterisation λ we choose for the curve. I.e., the length is function only of the curve between the two points and not of the parameterisation one uses along the curve.

Note that we will run into trouble if the expression for the length of the tangent vector is negative, since then the square root would be imaginary. In this case one must fudge things. One usually defines the length of a curve by taking the absolute value inside the square root, but this can run into trouble if the argument alters in sign along the curve. One almost always ignores such possibilities. One distinguishes the curves by the sign of the argument of the square root, and keeps curves with different signs separate.

0.4 Straight Lines

Now that we have a notion of length, we can discuss what we mean by a straight line. Euclid had the same problem, and he defined a straight line as the shortest distance between two points. While this often works, in special relativity, we know that for some straight lines (e.g., timelike curves) the straight line is the longest distance between two points.

Let us define a family of curves, $\gamma(\epsilon, \lambda)$ where the ϵ designates different curves between two points, which I will assume are always located at λ_1 and λ_2. Let the function $D(\epsilon)$ designate the distance between these two points along the various curves. We will say that the curve $\gamma(0, \lambda)$ is a straight line between the two points if for all sets of curves $\gamma(\epsilon, \lambda)$ such that $\gamma(0, \lambda)$ is that same curve, that $\frac{dD(\epsilon)}{d\epsilon}$ is zero. I.e., for all sets of curves, the given curve is at a relative minimum, maximum, or inflection point. Note we will always demand that the curves be nice curves.

Writing this in terms of coordinates, we have the expression

$$\frac{dD}{d\epsilon} = \int_{\lambda_1}^{\lambda_2} \frac{d}{d\epsilon} \sqrt{\sum_{ij} g_{ij}(x^k(\gamma(\epsilon, \lambda)) \frac{dx^i(\gamma(\epsilon, \lambda))}{d\lambda} \frac{dx^j(\gamma(\epsilon, \lambda))}{d\lambda}} d\lambda \tag{41}$$

Defining

$$S(\epsilon, \lambda) = \sqrt{\sum_{ij} g_{ij}(x^k(\gamma(\epsilon, \lambda)) \frac{dx^i(\gamma(\epsilon, \lambda))}{d\lambda} \frac{dx^j(\gamma(\epsilon, \lambda))}{d\lambda}} \tag{42}$$

we have

$$\int \frac{1}{2S} \quad \sum_{ij} \left(\sum_k \frac{\partial g_{ij}}{\partial x^k} \frac{dx^k}{d\epsilon} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right. \tag{43}$$

$$\left. + g_{ij} \frac{d^2 x^i}{d\epsilon d\lambda} \frac{dx^j}{d\lambda} + g_{ij} \frac{dx^i}{d\lambda} \frac{d^2 x^j}{d\epsilon d\lambda} \right) d\lambda \tag{44}$$

Since g_{ij} is symmetric, and since i, j, k are just "dummy" summation variables, we can rename them in the various terms to get

$$\int \frac{1}{2S} \sum_k \left(\sum_{ij} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^k}{d\epsilon} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right. \tag{45}$$

$$\left. + \sum_j 2 g_{kj} \frac{d^2 x^k}{d\epsilon d\lambda} \frac{dx^j}{d\lambda} \right) d\lambda \tag{46}$$

Integrating the second term by parts, and recalling that $\frac{dx^i}{d\epsilon}$ is zero at λ_1 and λ_2 (all the curves we are comparing are supposed to go through the same points at their endpoints), this expression becomes

$$\int \sum_k \frac{dx^k}{d\epsilon} \left(\frac{1}{2S} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^k}{d\epsilon} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right. \tag{47}$$

$$\left. - \frac{d}{d\lambda} \left[\frac{1}{S} g_{kj} \frac{dx^j}{d\lambda} \right] \right) d\lambda \tag{48}$$

Since we said that we wanted this to be zero for all sets of curves, the only way we can do this is if each term multiplying any $\frac{dx^k}{d\epsilon}$ for each value of k and for each point along the curve must be zero. Otherwise one can always choose a set of curves such that the integral is not zero. I.e., we get the second order differential equation

$$\frac{d}{d\lambda} \left(\frac{1}{S} \sum_j g_{kj} \frac{dx^j}{d\lambda} \right) = \frac{1}{2S} \sum_{ij} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \tag{49}$$

This is the geodesic equation.

This equation and the derivation can be simplified if we make a special choice for the parameter λ. Namely, if we choose λ to be such that $S = 1$, the S disappears from the above equation. This choice of the λ parameter is usually designated by s (or sometimes by τ. Furthermore, if we choose this parameterisation, then we have that

$$\left(\frac{d}{d\epsilon} \int S^n ds \right) |_{\epsilon=0} = \left(n \int S^{n-1} \frac{dS}{d\epsilon} ds \right) |_{\epsilon=0} = n \left(\int \frac{dS}{d\epsilon} ds \right) |_{\epsilon=0} = 0 \tag{50}$$

since along the solution curve $S = 1$. I.e., if we choose our parameter λ correctly (i.e.,, to be equal to the path length, s), we can place an arbitrary power of S in the integral and get the same equations of motion, and in particular we can use n=2 to get rid of the horrible square root, in the variation. This makes the equations much easier to derive, at the expense of not allowing an arbitrary parameterisation. Note that one has to choose this parameterization after the variation of the integral (deriviative by ϵ).

Thus, if we choose this parameterization, the geodesic equation becomes

$$\frac{d}{ds} \left(\sum_j g_{kj} \frac{dx^j}{ds} \right) = \sum_{ij} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{ds} \frac{dx^j}{ds} \tag{51}$$

with the additional requirement that

$$\sum_{ij} g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1 \tag{52}$$

It is easy to show that this second constraint equation is consistant with the second order equations above.

An example: Consider the metric

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{53}$$

Using the above we need to extremize the integral

$$\frac{d}{d\epsilon}\int (\frac{dr}{ds})^2 + r^2(\frac{d\theta}{ds})^2 ds = 0 \tag{54}$$

Let us first take the set of curves to be such that $\frac{d\theta}{d\epsilon}$=0. Then we have

$$\begin{aligned} 0 &= \int 2(\frac{dr}{ds})\frac{d^2r}{d\epsilon ds} + 2r\frac{dr}{d\epsilon}(\frac{d\theta}{ds})^2 ds & (55)\\ &= 2\int \frac{dr}{d\epsilon}(-\frac{d^2r}{ds^2} + r(\frac{d\theta}{ds})^2)ds & (56)\end{aligned}$$

and since $\frac{dr}{d\epsilon}$ is arbitrary (except at the end points which was why the endpoint contributions in the integration by parts disappeared), we must have as our first equation that

$$-\frac{d^2r}{ds^2} + r(\frac{d\theta}{ds})^2 = 0 \tag{57}$$

Now choosing the set of paths so that $\frac{dr}{d\epsilon}$ is zero, we get

$$\begin{aligned} 0 &= \int 2r^2\frac{d\theta}{ds}\frac{d^2\theta}{ds^2}ds & (58)\\ &= -2\int \frac{d\theta}{d\epsilon}(\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) & (59)\end{aligned}$$

which by the same reasoning on the arbitrariness of $\frac{d\theta}{d\epsilon}$ gives

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 0 \tag{60}$$

The third equation is

$$(\frac{dr}{ds})^2 + r^2(\frac{d\theta}{ds})^2 = 1 \tag{61}$$

The solution is

$$\frac{d\theta}{ds} = \frac{L}{r^2} \tag{62}$$

for some constant L and then

$$(\frac{dr}{ds})^2 = 1 - \frac{L^2}{r^2} \tag{63}$$

which gives

$$s = \int \frac{dr}{\sqrt{1 - \frac{L^2}{r^2}}} \tag{64}$$

$$s - s0 = \sqrt{r^2 - L^2} \tag{65}$$

or

$$r = \sqrt{(s - s_0)^2 + L^2} \tag{66}$$

Substituting into theequation for θ we have

$$\theta - \theta_0 = \int \frac{L}{(s - s_0)^2 + L^2} ds = atan(\frac{s - s_0}{L}) \tag{67}$$

Note that if we choose s_0 such that $\theta_0 = 0$, and define

$$x = r\cos(\theta), \qquad y = r\sin(\theta) \tag{68}$$

we have

$$x = L, \qquad y = s - s_0 \tag{69}$$

0.5 Inverse metric

We now have two different functions of a tangent vector which give a number. For any tangent vector V^A, the function of Z^B given by $f_{V^A}(Z^B) = g(V^A, Z^B)$. Similarly for any cotangent vector U_A we have the function $h_{U_A}(Z^B) = Z^A U_A$ is also a function from the set of vectors to the real numbers. Now, given a vector V^A one can always find a cotangent vector U_B such that $h_{U_A}() = f_{V^A}()$. To see this is most easily done using the components.

$$h_{U_A}(Z^B) = \sum_i U_i Z^i \tag{70}$$

$$f_{V^A}(Z^B) = \sum_i (\sum_j g_{ji} V^j) Z^i \tag{71}$$

I.e., if we choose

$$U_i = \sum_j g_{ij} V^j \tag{72}$$

we see that both functions f and h give the same value for all values of Z^A. I.e., the metric allows us to associate a unique cotangent vector for each tangent vector. Furthermore, it may allow us to associate a length to cotangent vectors. I.e., if U_A is assciated with V^A and W_B is associated with Z^B we can define a dot product $\tilde{g}(U_A, W_B) = g(V^A, Z^B)$ However, it is not true that this defines a metric for all cotangent vectors necessarily. For example, if the U_A associated with V^A is the null vector, when V^A is not a null vector, then there will be cotangent vectors which have no tangent vector as their source. The easiest case to see is if the metric g is zero for all arguments. Then clearly each tangent vector has the zero cotangent vector associted with it and for no non-zero cotangent vector is there any tangent vector.

In everything we do we will assume that this is not true, but rather that for each cotangent vector there is a unique, non-zero tangent vector which gives that cotangent vector via the metric. I.e., for each non-sero U^A there exists a unique V^A such that

$$U_i = \sum_j g_{ij} V^j \tag{73}$$

This means that there must be another set of numbers, which I will designate by g^{ij} such that if

$$U_i = \sum_j g_{ij} V^j \tag{74}$$

then

$$V^i = \sum_k g^{ik} U_k \tag{75}$$

This gives that for all vectors V^A, we have

$$V^i = \sum_{kj} g^{ik} g_{kj} V^k \tag{76}$$

or

$$\sum_{kj} g^{ik} g_{kj} = \delta^i_j \tag{77}$$

I.e., the matrix represented by the coefficients g^{ik} is the inverse matrix to the matrix g_{ik}.

Thus the tangent metric and the cotangent metric can be used to map tangent vectors to cotangent vectors or cotangent vectors to tangent vectors.

Consider a function $f(p)$ and a curve $\gamma(\lambda)$ such that $\gamma(\lambda)$ lies entirely withing the level surface of f. I.e., $f(\gamma(\lambda)) = f(\gamma(\lambda'))$ for all λ, λ'. Then $df_A \left(\frac{\partial}{\partial\gamma}\right)^A = 0$ and the tangent vector associated with df_A must be perpendicular (have zero dot product) with all of the tangent vectors which lie within the level surface of f. This is the usual gradient vector as an arrow that you have learned about in previous course. I.e., the gradient, as a cotangent vector is defined even in the most primative structure of the theory, but the association of a tangent vector (an arrow) with the gradient requires the existence of the metric.

We note that this can lead to some very strange situation. We will find that there exist metrics (e.g., the special relativisitic metric) such that a vector can be perpendicular to itself (i.e., have zero length). This means that the gradient vector, regarded as an arrow, can be a tangent vector which lies within the level surface itself. I.e., a tangent vector can both be tangent to the surface (i.e., to a curve which lies in the surface) and at the same time be perpendicular to the surface (have zero dot product with all tangent vectors to the surface).

0.6 Notation:

As with most physicists, I am lazy. I do not want to write additional symbolism when more compact will do. Thus if a have a function of coordinates $f(\{x^i\})$, instead of writing the partial derivative with respect to x^k as $\frac{\partial f}{\partial x^k}$, I will use the more compact notation

$$\partial_k f \equiv \frac{\partial f}{\partial x^k} \tag{78}$$

And sometimes I will use an even more compact notiation

$$f_{,k} \equiv \partial_k f \tag{79}$$

this being even simpler to write. Of course it can also be confusing if you are not used to the notation.

For the metric, if we define the length of a curve by $s(\lambda)$ we know that the length of a tangent vector with components $\frac{dx^i}{d\lambda}$ as

$$(\frac{ds}{d\lambda})^2 = \sum_{ij} g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \tag{80}$$

To specify the metric we could write out the above in detail. Since the value of λ is irrelevant, one often simply removes all of the $d\lambda$ and writes the metric as

$$ds^2 = \sum_{ij} g_{ij}dx^i dx^j \tag{81}$$

One thing to be careful of is to remember that since the metric is symmetric, there will be two terms multiplying each $dx^i dx^j$ if i and j are not equal. Thus

$$ds^2 = (1-\frac{2M}{r})du^2 + 2dudr - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{82}$$

has components of the metric given by

$$g_{uu} = (1-\frac{2M}{r}) \tag{83}$$

$$g_{ur} = g_{ru} = 1 \tag{84}$$

$$g_{\theta\theta} = r^2 \tag{85}$$

$$g_{\phi\phi} = r^2 sin(\theta)^2 \tag{86}$$

and all the rest of the components being 0. Note that g_{ur} is 1, not 2. Einstein himself messed up in one of his notebooks, and confused himself for a year (thinking that the flat spacetime metric in rotating coordinates was not a solution of his equations) because he forgot this.