Physics 501-25
Accelerated Detector

In special relativity, a detector with a constant acceleration follows a path given by

$$t = \frac{1}{a}\sinh(a\tau) \tag{1}$$

$$x = \frac{1}{a}\cosh(a\tau) \tag{2}$$

Here a is the acceleration and τ is the proper time along the trajectory of the detector. (we have $dt = cosh(a\tau)d\tau;\quad dx = sin(a\tau)d\tau$ for small $d\tau$. Then $dt^2 - dx^2 = (cosh^2(a\tau) - sinh(a\tau)^2)d\tau^2 = d\tau^2$, which is just the expression for the proper time along the path.

Note that for τ near 0,

$$t \approx \tau \tag{3}$$

$$x \approx \frac{1}{a} + \frac{1}{2}a\tau^2 = \frac{1}{a} + \frac{1}{2}at^2 \tag{4}$$

which is just the equation for an accelerated object.

(Note that I am using units in which $c = 1$ just as I used units in the quantum parts so that $\hbar = 1$.)

The first thing is that for the detector, it is the proper time, not the time t which determines its internal dynamics.Thus for the two level system, the equation of motion for the annihilation operator of the two level system in the Heisenberg equations of motion will be

$$\sigma_- = \sigma_{0-}e^{-iE\tau} = \sigma_{0-}e^{-i(E/a)\text{arcsinh}(at)} \tag{5}$$

Secondly, the trajectory of the detector is $x_0 = \sqrt{\frac{1}{a^2} - t^2}$. which is a hyperbola in x t coordinates with assymptotes being the two null lines $t \pm x = 0$.

Thus if we have such an accelerated detector, the interaction Hamiltonian will be affect the state of the detector and of the field(given again that $|\psi, 0\rangle = |\phi\rangle\,|{\downarrow}\rangle$)

$$\int H_I\,|\phi\rangle\,|{\downarrow}\rangle = \epsilon\left[\int \sigma_0^\dagger e^{iE\tau(t')}\sum_i\left(A_i\partial_t'\phi_i(t', x(t')) + A_i^\dagger\partial_t'\phi_i^*(t', x(t'))\right)dt'\right] \tag{6}$$

The Quantum field without interaction is

$$\Phi_0(t, x) = \sum_{\pm}\int_0^\infty \frac{A_{\omega\pm}}{\sqrt{2\pi 2\omega}}e^{i\omega t \pm x} + HermConj. \tag{7}$$

In the case where the detector was at rest, the integral over t' in the interaction picked out the $A_E\sigma_0^\dagger$ amd $A_E^\dagger\sigma_0$ terms. If the detector is moving however, both the temporal and spatial parts of the modes are important since both change as the detector moves.

Because the detector has proper time dependence, we can switch our integration to make the integration variable be τ rather than t.

We have $dt' = \frac{dt'}{d\tau} d\tau$ Also

$$\partial_{t'}\phi(t, x(t')) = \frac{d\tau}{dt'}\partial_\tau \phi(t(\tau), x(\tau)) = \frac{1}{\frac{dt'}{d\tau}}\partial_\tau \phi(t(\tau), x(\tau)) \tag{8}$$

and thus

$$\partial_{t'}\phi(t, x(t'))dt' = \partial_\tau \phi(t(\tau), x(\tau))d\tau \tag{9}$$

It would be really easy if we could choose our modes $\phi_i(t(\tau), x(\tau))$ such that they went like $e^{i\omega\tau}$, but were still made up solely of temporal Fourier modes with temporal dependence $e^{i\omega t}$,since then the integral over τ would be easy. Fortunately such modes exist.

I will here restrict myself to 1+1 dimensions, and to a massless ($m = 0$) field. While the calculations are far easier there, they can be carried out almost as easily for a massive field theory and in higher than 1 spatial dimensions.

What we would like is to have the modes ϕ_i go as $e^{-i\nu\tau}$. Let us look at the solutions of the field equations.

$$\partial_t^2\phi - \partial_x^2\phi = 0. \tag{10}$$

we can solve this with Fourier modes $e^{-i(\omega t - kx)}$ with $\omega = |k|$ One thus has the solutions $e^{-i|k|(t-x)}$ or $e^{i|k|(t+x)}$. Writing t and x in terms of τ we have these solutions as

$$e^{-i|k|t(\tau)-x(\tau)} = e^{-i(|k|(-e^{-a\tau})/a} \tag{11}$$
$$e^{-i|k|(t(\tau)+x(\tau)} = e^{-i(|k|/a(e^{a\tau})/a} \tag{12}$$

Which is a bit of mess. We can certainly do the required integral, but there is an easier way.

Let us define a new coordinate system, $\tau,\ \rho$ where

$$t = \frac{1}{a}\sinh(a\tau)e^{a\rho} \tag{13}$$
$$x = \frac{1}{a}\cosh(a\tau)e^{a\rho} \tag{14}$$

where $\rho = 0$ is the path of the detector.

Then we havethe metric given by

$$ds^2 = dt^2 - dx^2 = (\cosh(a\tau)e^{a\rho}d\tau + \sinh(a\tau)e^{a\rho}d\rho)^2 - (\cosh(a\tau)e^{a\rho}d\rho + \sinh(a\tau)e^{a\rho}d\tau)^2 \tag{15}$$
$$= e^{2a\rho}(d\tau^2 - d\rho^2) \tag{16}$$

One problem is that for all ρ, $e^i a\rho$ is positive, so this new set of coordinates cover just the positive values of x. To also cover the negative values of x, define another coordinate ρ' so that

$$t = \frac{1}{a}\sinh(a\tau)e^{-a\rho'} \tag{17}$$

$$x = -\frac{1}{a}\cosh(a\tau)e^{-a\rho'} \tag{18}$$

These coordinates are a version of what are called Rindler coordinates, after Wolfgang Rindler, a physicist who died a few years ago, and was responsible for many of the "paradoxes" that you have studied in your special relativity course. It was named that by Steve Fulling, who, after having studied Parker's cosmological quantisation papers, got interested in flat spacetime. Einstein and Rosen, in a footnote of their famous wormhole paper, had showed that a coordinate change like the above was possible in flat spacetime. Rindler had rediscovered it, and emphasized the similarity of this set of flat spacetime coordinates to the Schwartzschild coordinates used by Schwartzschild in his solution to Einstein's gravitational theory, what we now call a black hole solution. The $\rho = -\infty, \tau = \pm\infty$ surface, $(t \pm x = 0)$, is similar to the horizon of a black hole, $r = 2GM/c^2, t = \pm infty$, surfaces. $(e^{a\rho} \to r - 2GM)$.

Note that $\partial_\tau t)$ is positive for values of τ and ρ, ρ', and that $\partial_\rho x$ and $\partial rho' x$ are also always positive for all values of τ and ρ, rho'.

Let us now go to null coordinates. Define

$$U = t - x; \quad V = t + x \tag{19}$$

$$u = \tau - \rho; \quad v = \tau + \rho \tag{20}$$

$$u' = \tau - \rho'; \quad v = \tau + \rho' \tag{21}$$

Then we have

$$U = -\frac{1}{a}e^{-au}; \quad U = \frac{1}{a}e^{au'} \tag{22}$$

$$V = \frac{1}{a}E^{au}; \quad V = -\frac{1}{a}e^{-au'} \tag{23}$$

While u, v are only defined for $x > 0$ and $|t| < x$ and u', v' are only defined for $x < 0$ and $|t| < -x$, U and V are defined for all values of U, V Ie, Minkowski flat space null coordinates are an analytic extention of the Either set of Rindler coordinates for the right(+) or the left (-) regions. They are also defined so that all increase for increasing t

The equations of motion for the field are given by

$$0 = \partial_U\partial_V\Phi(U,V) = \partial_u\partial_v\Phi(u,v) = \partial_u\partial_v\Phi(u',v') \tag{24}$$

A set of solutions are therefor given by

$$\Phi(U,V) = \int_0^\infty \frac{\tilde{A}_{\omega-}}{\sqrt{4\pi|\omega|}}e^{-i\omega U} + \frac{A_{\omega+}}{\sqrt{4\pi|\omega|}}e^{-i\omega V}d\omega + HermConj \tag{25}$$

Similarly using the u, v and u', v' coordinates, we have solutions.

$$\Phi(u,v) = \int_0^\infty \frac{A_{\Omega-}}{\sqrt{4\pi|\Omega|}} e^{-i\omega u} + \frac{A_{\Omega+}}{\sqrt{4\pi|\omega|}} e^{-i\Omega v} d\Omega + HermConj \tag{26}$$

$$\Phi(u',v') = \int_0^\infty \frac{A_{\Omega-}}{\sqrt{4\pi|\Omega|}} e^{-i\omega u'} + \frac{A_{\Omega+}}{\sqrt{4\pi|\omega|}} e^{-i\Omega v'} d\Omega + HermConj \tag{27}$$

These two are of course only defined where u, v or u', v' are defined, namely for $x > 0$ or $x < 0$ and $|t| < |x|$.

Note that in each case,

$$\partial_t U = \partial_t(V) = \partial_\tau u = \partial_\tau v = \partial_\tau u' = \partial_\tau v' = 1 \tag{28}$$

Thus we have that

$$< \phi_1 v, \phi_2(v) = i \int \left[\phi_1^*(v)\partial_t \phi_2 * (v) - \phi_2(v)\partial_t \phi_1^*(v)\right] dx \tag{29}$$

$$= i \int \left[\phi_1^*(v)\partial_t \phi_2 * (v) - \phi_2(v)\partial_v \phi_1^*(v)\right] dv \tag{30}$$

where v stands for any of U, V, u, v, u', v'.

Furthermore since the norm is the temporal part of conserved vector,

$$J_k(\psi_1, \psi_2) = i(\psi_1^* \partial_k \psi_2 - \partial_k \psi_1^* \psi_2) \tag{31}$$

it does not matter what hypersurface the integral is evaluated over. (This is just Gausses law). We can then take the hypersurface to be the null V=0 hypersurface.

This leads to the result that the normalisation factor needed to make the modes orthonormal, is just that given above.

iThere is a theorem that if one mades a function f(t) out of only positive frequencies $f = \int_0^\infty alpha_\omega e^{-i\omega t} d\omega$, then f must be analytic for $Im(t) < 0$. $(e^{-i\omega t} = e^{-i\omega Re(t)} e^{-omegaIm(t)}$ which goes to 0 for large ω.

Let us look at the mode

$$e^{-i\Omega v} = (aV)^{i\Omega/a} \tag{32}$$

This is well defined, except at $U = 0$ which has a branch cut. we can decide in which direction to take that branch cut. Let us take it to extend into the upper U plane, leaving a function which it analytic in thelower half U plane. But a function which is analytic in the lower half U plane can be constructed out of fourier modes which go as $e^{-i\omega U}$ with positive ω. Thus choosing the branch cut in that way, we get a function which is created out of positive norm Minkowski fourier modes.

Let us look at the above modes. For the V modes, since $V = t + x$, as V increases, so does t (as long as say $x(t)$ is timelike).

The problem with this function is at $V = 0$ where the function has a singularity. We can make it analytic in the upper $Im(V)$ by deforming the integral path of V into the lower plane Ie, $\lim_{\lambda\to+0}(V+i\lambda)^{\pm i\frac{|\kappa|}{a}}$ is analytic in the upper half $Im(V)$ half plane and is made up of Fourier components $e^{-i\omega V}$. This is true independent of the sign of $\pm|\kappa|$.

Let us look at the functions

$$\phi_\Omega(V) = (V+i\lambda)^{i\Omega/a} \tag{33}$$
$$\phi_\Omega(U) = (U+i\lambda)^{i\Omega/a} \tag{34}$$

where Ω is real but of arbitrary sign. The $i\lambda$ notation indicates that λ is positive, and that one takes the limit of λ goes to 0 from the positive direction. The singularity occurs at $V = -i\lambda$ and as a result the functions are analytic and bounded in the whole positive imaginary V plane for all real Ω. The norm is

$$\phi_\Omega, \phi'_\omega >= i\int(\phi^*_\Omega\partial_t\phi'_\Omega - \phi'_\Omega\partial_t\phi^*_\Omega)dx \tag{35}$$
$$= i\int(\phi^*_\Omega\partial_V\phi'_\Omega - \phi'_\Omega\partial_V\phi^*_\Omega)dV \tag{36}$$

since $\partial_t V = \partial_x V = 1$.

$$\phi_\Omega, \phi'_\omega >= i\int(\phi^*_\Omega\partial_t\phi'_\Omega - \phi'_\Omega\partial_t\phi^*_\Omega)dx \tag{37}$$
$$= i\int(\phi^*_\Omega\partial_v\phi'_\Omega - \phi'_\Omega\partial_v\phi^*_\Omega)dv + \int(\phi^*_\Omega\partial_v\phi'_\Omega - \phi'_\Omega\partial_{\tilde v}\phi^*_\Omega)d\tilde v \tag{38}$$

where $v = \tau+\rho$ and $\tilde v = \tau - \rho'$. Now,

$$\phi_\Omega(V + i(\lambda = +0)) = V^{i\Omega/a}\theta(V) + e^{-\pi\Omega/a}(-V)^{i\Omega/a} \tag{39}$$

since near 0 the phase of V goes from 0 radians to π radians as V goes from positive to negative values of V. Thus the phase of V, the imaginary part of $ln(V)$ goes from 0 to π as V goes from positive to negative values. Since the phase is multiplied by $i\Omega/a$, we get that the amplitude for negative values of V is smaller than positive by $e^{-\pi\Omega/a}$. If Ω is positive, then the amplitude for negative V is exponentially smaller than for positive V. If Ω is negative, then negative amplitudes are exponentially larger than for positive V.

We can now evaluate the norm.

$$< \phi_\Omega \quad , \quad \phi'_\Omega >$$
$$= \int_0^\infty (aV)^{-i\Omega/a}\partial_V(aV)^{i\Omega'/a} - \partial_V(aV)^{-i\Omega/a}(aV)^{i\Omega'/a})dV \tag{40}$$
$$+ \int_{-\infty}^0 e^{-\pi(\Omega+\Omega')/a}(a|V|)^{-i\Omega/a}\partial_V(a|V|)^{i\Omega'/a} - \partial_V(a|V|)^{-i\Omega/a}(a|V|)^{i\Omega'/a}dV \tag{41}$$
$$= i(1-e^{-\pi(\Omega+\Omega')/a}a^{i(\Omega'-\Omega)}i(\Omega+\Omega')\int_0^\infty \frac{|V|^{i(\Omega-\Omega')/a}}{|V|}d|V| \tag{42}$$
$$\tag{43}$$

But

$$\int_0^\infty |V|^{i(\Omega-\Omega')/a}\frac{d|V|}{|V|} = \int_{-\infty}^\infty e^{i(\Omega-\Omega')\zeta/a}d\zeta = 2\pi\delta((\Omega-\Omega')/a) \tag{44}$$

where $|V| = e^\zeta$.

Thus

$$<\phi_\Omega, \phi_{\Omega'}>= 2\pi a\delta(\Omega-\Omega')(1-e^{-\pi(2\Omega)/a} = 2\pi\delta(\Omega-\Omega')\frac{\sinh(\pi\Omega/a)}{e^{-\pi\Omega/a}} \tag{45}$$

Thus the Normalisation factor for these modes is $\frac{N=e^{\pi}\Omega/(2a)}{\sqrt{(2\Omega\sinh(\pi\Omega/a)}}$. (since both $\Omega$and $\sinh(\pi\Omega/a)$ are odd functions of Ω, the quantity under the square root is always positive)

For positive Ω, this mode is concentrated in the right Rindler Wedge ($V > 0$). for negative Ω it is concentrated in the left wedge.

One can go through exactly the same procedure for the U modes. One gets

$$\phi_\Omega = \frac{1}{\sqrt{2\Omega\sinh(\pi\Omega/a)}}(e^{\pi\Omega/(2a)}\theta(U) + e^{-\pi\Omega/(2a)}\theta(-U))|aU|^{-i\Omega/a} \tag{46}$$

so again for positive Ω the mode is dominant in the sector $U > 0$ and is smallest in the $U < 0$ sector. For $\Omega < 0$ the opposite is again true.

Since each of these positive norm modes can be written in terms of the Minkowski positive norm "Hamiltonian-diagonalisation" modes, the annihilation operators of these modes will be linear combinations of the Minkowski "Hamiltonian diagonalisation" Annihilation operators and have the same vacuum state $|0\rangle$. So, let us choose our ϕ_i to be these modes.

$$\begin{aligned}\Phi &= \int_{-\infty}^\infty \frac{e^{\pi\Omega/2a}}{\sqrt{2\Omega\sinh(\pi\Omega/2a)2\pi}}\left[A_{\Omega v}(e^{-i\Omega(\tau+\rho)} + A_{-\Omega u}e^{i\Omega(\tau-\rho)})\right] + HC; \qquad x > 0 && (47)\\ &= \int_{-\infty}^\infty \frac{e^{\pi\Omega/2a}}{\sqrt{2\Omega\sinh(\pi\Omega/2a)2\pi}}\left[A_{-\Omega v}(e^{-i\Omega(\tau+\rho')} + A_{\Omega u}e^{i\Omega(\tau-\rho')})\right] + HC; \qquad x < 0 && (48)\end{aligned}$$

The state defined by

$$A_{\Omega u}|0\rangle = A_{\Omega v}|0\rangle = 0 \tag{49}$$

is exactly the same as the vacuum state defined by the usual Minkowski Annihilation operators. Note that these states are defined for all Ω, not just positive values. This is another example that "positive frequency" is NOT a sensible criterion for defining the modes corresponding to annihilation and creation operators.

This can be inserted into the expression for the first order amplitude for the detector. Since the detector lives solely in the region x¿0, we only need the expression for the detector in the right hand wedge Let us choose the state to be the Minkowski vacuum state. This is the state where particle detectors at rest seen nothing. Choosing the initial state of the field to be the Minkowski vacuum state, the annihilation operators all give 0 on that vacuum state. The detector is located at $\rho = 0$. If T is large then the integral will pick out Ω only near -E.

$$|\psi, 0\rangle = |0\rangle |\downarrow\rangle \tag{50}$$

$$|\delta\psi, T\rangle \approx -i\epsilon E \frac{e^{-\pi E/2a}}{\sqrt{2\pi E \sinh(\pi E/a)}} \int_0^T e^{iE\tau} e^{-i\Omega\tau} A^\dagger_{-\Omega} |0\rangle \, d\Omega d\tau \tag{51}$$

The probability of detection will be

$$P_\uparrow = \langle \delta\psi, T| |\uparrow\rangle \langle\uparrow| |\delta\psi, T\rangle \tag{52}$$

$$= \epsilon^2 \frac{e^{-2\pi E/a}}{(1 - e^{-2\pi E/a})} \frac{E}{4\pi} \int | \int_0^T e^{i(E-\Omega)\tau} d\tau|^2 d\Omega \tag{53}$$

The first term is expected as the probability should grow as ϵ^2. The second term is just the Einstein-Bose thermal factor with temperature of $\frac{a}{2\pi}$. The third is

$$| \int_0^T e^{i(E-\Omega)\tau} d\tau|^2 = \left(\frac{sin((E-\Omega)T/2)}{E-\Omega} \right)^2 \tag{54}$$

and

$$\int \left(\frac{sin((E-\Omega)T/2)}{E-\Omega} \right)^2 d\Omega = \pi T/2 \tag{55}$$

Ie, the probability grows linearly in time, which is what one would expect of a random excitation probability.

The detector is excited at a constant rate, and with a factor that is just the thermal factor times a "cross section" for detection in flat spacetime.