Physics 530-21
Assignment 2

1. Given a function $f(p)$ and a set of coordinates $x^i(p)$ show that the two functions

$$f(p) \tag{1}$$

$$\sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}(x^i(p) - x^i(p_0)) \tag{2}$$

have the same cotangent vector at the point p_0. (p(x) is the point p in the space corresponding to the coordinates $\mathbf{x}$. Those partial derivatives are evaluated at the point p_0. Since $\frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}$ are constants, by the definition of the sum of cotangent vectors, this means that

$$df_A = \sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0} dx^i_A \tag{3}$$

Let

$$f(\gamma(\lambda)) = f(p(x^i(\gamma(\lambda)))) \tag{4}$$

Then

$$\frac{df}{d\lambda} = \sum_i \partial_i f(p(x^i(\gamma(\lambda))))\frac{dx^i(\gamma(\lambda))}{d\lambda} \tag{5}$$

But the other function has

$$\frac{d}{d\lambda}\sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}(x^i(\gamma(\lambda)) - x^i(p_0)) = \sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}\frac{dx^i(\gamma(\lambda))}{d\lambda} \tag{6}$$

which at the point $p = p_0$ is the same.

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2. Show that if x^i and $\tilde{x}^i$ are two different coordinates , and $\gamma(\lambda)$ and $\gamma'(\lambda)$ are two different curves through the point p_0 with the point p_0 corresponding to the same value, $\lambda = 0$ in both cases, that the two curves defined by

$$\Gamma(\lambda) = p(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0)) \tag{7}$$

$$\tilde{\Gamma}(\lambda) = p(\tilde{x}^i(\gamma(\lambda)) + \tilde{x}^i(\gamma'(\lambda)) - \tilde{x}^i(p_0)) \tag{8}$$

have the same tangent vector at the point p_0

$$\frac{d}{d\lambda} f(\Gamma(\lambda_)) = \frac{d}{d\lambda} f(p(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0))) \quad (9)$$

$$= \frac{d}{d\lambda} f(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0)) \quad (10)$$

$$= \frac{\partial}{\partial x^i}_f (x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0)) \frac{d}{d\lambda}(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0) \quad (11)$$

$$= \frac{\partial}{\partial x^i}_f (p_0)(\frac{dx^i}{d\lambda} + \frac{dx'^i}{d\lambda)}) \quad (12)$$

$$= \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial f}{\partial \tilde{x}^j} (p0)(\frac{d\tilde{x}^j}{d\lambda} \partial \frac{x^i}{\partial x^j} + \frac{d\tilde{x'}^j}{d\lambda} \partial \frac{x^i}{\partial x^j}) \quad (13)$$

But $\frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^i} = \delta^k_i$ which shows that the two are the same for all functions f.
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This shows that while the definition of the addition of two tangent vectors is defined via coordinates, the sum tangent vector thus defined does not depend on which coordinates we use, although the curves Γ and $\tilde{\Gamma}$ are in general different.

As an example, consider the two curves in two dimensions with coordinates x,y and r, θ

$$\gamma : \quad (14)$$

$$x = \lambda \quad (15)$$

$$y = 1 \quad (16)$$

$$\gamma' : \quad (17)$$

$$y = 1 + 2\lambda \quad (18)$$

$$x = 0 \quad (19)$$

Now write those same two curves in terms of the coordinates r, θ where

$$x = r\cos(theta)y = r\sin(theta) \quad (20)$$

Show that the sum curve $\Gamma(\lambda)$ defined in the two coordinate systems differ, but that at the point $\lambda = 0$ their tangent vectors do not.

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The second components are

$$r\cos(\theta) = \lambda \quad (21)$$

$$r\sin(\theta) = 1 \quad (22)$$

$$r'\cos(\theta') = 0 \quad (23)$$

$$r'\sin(\theta') = 1 + 2\lambda \quad (24)$$

or

$$r = \sqrt{1 + \lambda^2} \quad (25)$$

$$tan(\theta) = atan(1/\lambda) \quad (26)$$
$$r' = 1 + 2\lambda \quad (27)$$
$$\theta' = \pi/2 \quad (28)$$

so in the first case the sum is

$$x_s = 1 + 3\lambda \quad (29)$$
$$y_s = 1 + 2\lambda \quad (30)$$
$$r_s = \sqrt{1+\lambda^2} + 1 + 2\lambda - 1 \quad (31)$$
$$\theta_s = atan(1/\lambda) + \pi/2 - \pi/2 \quad (32)$$

Then

$$\tilde{f}(r,\theta) = f(r\cos(\theta), r\sin(\theta)) \quad (33)$$
$$\frac{d}{d\lambda}(\tilde{f}(r_s,\theta_s) \quad (34)$$
$$= f((\sqrt{1+\lambda^2} + 2\lambda)cos(atan(1/\lambda)), (\sqrt{1+\lambda^2} + 2\lambda)sin(atan(1/\lambda))) \quad (35)$$

We are going to take the derivative at $\lambda = 0$ so all we need is the first order approximation for r_s and θ_s

$$r_s \approx 1 + 2\lambda \quad (36)$$
$$cos(atan(1\lambda)) = \sqrt{1/(\sqrt{1+1/lamda^2})} \approx \lambda \quad (37)$$
$$sin(atan(1\lambda)) = \sqrt{1 - \frac{\lambda^2}{(1+\lambda^2)}} \approx 1 \quad (38)$$

so

$$\frac{d}{d\lambda}\tilde{f} = \frac{d}{d\lambda}f((1+2\lambda)\lambda, (1+2\lambda)) = \frac{d}{d\lambda}(f(\lambda, 1+2\lambda) \approx f(x_s(\lambda), y_s(\lambda) \quad (39)$$

Ie, the two derivatives are the same at $\lambda = 0$
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3. Assume that ${H^A}_B$, ${{L_A}^B}_C$ and M_{AB} are tensors, and f, g are functions. Which of the following are tensors and why?
i)${Q_A}^B = {H^B}_A$

Yes. Left is function of tangent vector assoc with A and cotangent with B, while rhs has tangent associated with A and cotang with B.
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ii)$R = {H^A}_A$

Yes, contraction on right to give scalar while left is a scalar.

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iii)$T^D_{ABC} = H^D{}_A M_{BC}$

Yes, both sides are function of three tangent and one cotangent vectors, and are linear.
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iv)$T^D_{ABC} = H^D{}_A + M_{BC}$

No. Not linear on right.
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v) $R^A = L_B{}^A{}_B$

No. Different arguments on left and right. On right B is not a contraction, but designates a quadratic function of a tangent vector. Not linear.
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vi) $S_A = L_A{}^B{}_B - L_B{}^B{}_A$

Yes, On left function of one tangent vector, and same on right. The B indices indicate contraction, not function.
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vii) $T_A = \nabla_B H^B{}_A$

Yes. Right is contraction on B, and single linear function of tangent vector.
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Is $\partial_i H^j{}_k$ the component of a tensor?

No. Unless in special coordinate sysem where Γ are all zero.
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What are the components of the expression below expressed in terms of partial derivatives and Christofel symbols?

$$\nabla_A H^A{}_B \tag{40}$$

$\partial_i H^i{}_j + \Gamma^i_{ki} H^k{}_j - \Gamma^k_{ji} H^i{}_k$
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4. Given coordinates r, θ, what are the tangent vectors to the curves defined by the coordinate conditions expressed in terms of $\frac{\partial}{\partial r}^A$ and $\frac{\partial}{\partial \theta}^A$

$$r(\lambda) = r_0 \tag{41}$$
$$\theta(\lambda) = \lambda \tag{42}$$

$$0\frac{\partial}{\partial r}^A + 1\frac{\partial}{\partial \theta}^A$$

Lengths a) 1 b)r

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$$r(\lambda) = \lambda \tag{43}$$
$$\theta(\lambda) = 5 * \lambda \tag{44}$$

$$1\frac{\partial}{\partial r}^{A} + 5\frac{\partial}{\partial \theta}^{A}$$

Lengths a) $\sqrt{26}$ b) $\sqrt{1+25r^2}$

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$$r(\lambda) = 10\lambda \tag{45}$$
$$\theta(\lambda) = 50 * \lambda \tag{46}$$

$$10\frac{\partial}{\partial r}^{A} + 50\frac{\partial}{\partial \theta}^{A}$$

Lengths a) $10\sqrt{26}$ b) $10\sqrt{1+24r^2}$

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What is the cotangent vector of the following functions

$$f(r,\theta) = r^2 \tag{47}$$

$$df_A = 2r dr_A + 0 d\theta_A$$

Lengths a) r^2 b) r^2

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$$f(r,\theta) = r^2 + \theta^2 \tag{48}$$

$$df_A = 2r dr_A + 2\theta d\theta_A$$

Lengths a) $\sqrt{4r^2 + 4\theta^2}$ b) $\sqrt{4r^2 + 4\frac{\theta^2}{r^2}}$

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In each case find the lengths of these various vectors for each point at which they are defined if the metric is given by a)

$$ds^2 = dr^2 + d\theta^2 \tag{49}$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{50}$$

5) Giventhe metric

$$ds^2 = \rho^2 dt^2 - d\rho^2 \tag{51}$$

solve forthe geodesicsof thismetricfor therelationsbetweenρ, $t$and$s$

This is very similar to the problem done in class except for the change in sign and the change in the names of the coordinates.

$$I = \int \rho^2 \frac{dt}{ds}^2 - \frac{d\rho}{ds}^2 ds$$

Varying with respect to t, we get

$$\int 2\rho^2 \frac{dt}{ds}\frac{d\delta t}{ds} ds = \int \frac{d}{ds}\left(2\rho^2 \frac{dt}{ds}\frac{d\delta t}{ds)ds - \frac{d}{ds}(2\rho^2 \frac{dt}{ds})\delta t}\right) \tag{52}$$

The first term is a complete integral and is evaluated on the boundaries . Assime δt is zero on the boundaries. Since for an arbitary variation this variation is supposed to be zero, we get

$$\frac{d}{ds}(2\rho^2 \frac{dt}{ds} = 0$$

or

$$(2\rho^2 \frac{dt}{ds}) = const. = E$$

We have one more variable ρ abd we could either do the variation or use the fact that s is defined such that

$$\rho^2 \frac{dt}{ds}^2 - \frac{d\rho}{ds}^2 = \{\pm 1, 0\} \tag{53}$$

$$(\frac{d\rho}{ds})^2 = \frac{E^2}{\rho^2} - \{\pm 1, 0\} \tag{54}$$

$$\frac{\frac{d\rho}{ds}}{\sqrt{\frac{E^2}{\rho^2} - \{\pm 1, 0\}}} = 0 \tag{55}$$

Thus

$$\int \frac{\rho}{\sqrt{E^2 - \{\pm\rho^2, 0\}}} d\rho = s \tag{56}$$

For a timelike geodesic This gives

$$\sqrt{E^2 - \rho^2} = s \tag{57}$$

$$\rho^2 = E^2 - s^2 \tag{58}$$

and

$$t = \int \frac{1}{E^2 - s^2/ds} = ln(\frac{E + s/2}{E - s/2}) \tag{59}$$

Ie, the timelike geodesics have only a finite range of s and have t go from $-\infty,\ \infty$ while ρ goes comes out of $\rho = 0$ and goes back to $\rho = 0$

For a spacelike curve, we have

$$\frac{d\rho}{ds} = \sqrt{E^2 + \rho^2}/\rho \tag{60}$$

The simplest case is to take $E = 0$, in which case $\rho = 2s$ and $t = \text{const}$.

For $E \neq 0$, we get

$$E^2 + \rho^2 = s^2 \tag{61}$$

$$\rho^2 = s^2 - E^2 \tag{62}$$

Again ρ must start out as 0, but it increases to infinity as s goes to infinity. Then

$$t = ln((s + E)/s - E)) \tag{63}$$

which again goes to -infinity as s goes to E or goes to infinity as s goes to infinity.

In fact thse coordinates are simply another set of coordinates for flat spacetime. Let T, Z be the usual coordinates for flat spacetime

$$ds^2 = dT^2 - dZ^2 \tag{64}$$

Now define t, ρ as

$$T = e^\rho sinh(t); \qquad Z = e^\rho cosh(t) \tag{65}$$

Then this gives the above metric. Clearly the geodesics of the T, Z metric are easy.