Physics 530-23
Assignment 2 Solutions

1) Linearized Equations:
Consider the metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \beta\frac{2M}{r})(dx^2 + dy^2 + dz^2) \quad (1)$$

find the deflection of light as a funtion of β. (Keep terms only to lowest order in M)

(The theory with $\beta = 0$ was Einstein's original theory of gravity, while $\beta = -1$ corresponds to the Nordstom theory)

Hint: You could do a sucsessive approximation by finding the path of a light ray travelling in the x direction, along z=0, y=b for M=0. Find the equation for the change in the tangent vector to the real path by integrating along the above path. The angular change in direction of the tangent vector will give the deflection angle.

One firstly needs to write down the geodesic equation in this metric

$$\frac{d}{ds}((1 - \frac{2M}{r})\frac{dt}{ds}) = 0 \quad (2)$$

$$-2M\frac{x}{r^3}\frac{dt}{ds} - \beta 2M\frac{x}{r^3}((\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 + (\frac{dz}{ds})^2) - 2\frac{d}{ds}((1 + \beta\frac{M}{r}\frac{dx}{ds}) = 0 \quad (3)$$

and similarly for y and z. Now let us assume that we start with the light ray traveling in x direction, with $z = 0, y = b$, and the initial $x = -\infty$. We take $\frac{dt}{ds} = \frac{dx}{ds} = 1$ initi/opt/zoom/ZoomLauncherally. If M were 0 then this would be true for the whole trajectory. If M is not zero, but $\frac{M}{b}$ is very small, then the change in $\frac{dx}{ds}$ and $\frac{dt}{ds}$ will be small ($<< \frac{2H}{b}$). However although $\frac{dx}{ds}$ and $\frac{dt}{ds}$ both stay very near 1, $\frac{dy}{ds}$ will be small. The deflection angle will then by $\frac{dy/ds}{dx/ds} \approx \frac{dy}{ds}$.,

The equation for $\frac{dy}{ds}$ is

$$2\frac{d}{ds}\frac{dy}{ds} \approx -2M\frac{y}{r^3} - \beta 2M\frac{y}{r^3} \quad (4)$$

All of the rest of the terms will be of quadratic order in M. Thus

$$\frac{d}{ds}\frac{dy}{ds} \approx -M(1 + \beta)\frac{b}{\sqrt{x(s)^2 + b^2}^3)} \quad (5)$$

where to lowest order $x(s) = +s$ so

$$\Delta\frac{dy}{ds} \approx -M(1 + \beta)\int_{-\infty}^{\infty} \frac{b}{(s^2 + b^2)^{3/2}}ds \quad (6)$$

$$= M(1 + \beta)\frac{1}{b}\int_{-\infty}^{\infty} \frac{1}{((s/b)^2 + 1)^{3/2}}d(s/b) = \frac{2M}{b}(1 + \beta) \quad (7)$$

For $\beta = 1$ we get Einstein's result. For $\beta = 0$ we get what has been called the Newtonian result (Einstein's 1908 result, where it was just g_{tt} that he changed), and for $\beta = -1$ the Nordsstom result, no defection. Fortunately Froelich's attempt to measure in the 1914 eclipse in the Ukraine failed.

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2)

$$\tilde{S}^i{}_j(\tilde{x}(x)) = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j} S^k{}_l(x) \tag{8}$$

Show that if $\tilde{x}^i = x^i + \xi^i(x)$. Find the change in the components of S to lowest order in ξ.

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We have essentially gone through this in class where S_{ij} is the metric

$$\tilde{S}_{kl} = \frac{\partial_{x^i}}{\partial_{\tilde{x}^k}} \frac{\partial_{x^j}}{\partial_{\tilde{x}^l}} S_{ij} \tag{9}$$

$$\approx (\delta^i_k + \partial_k \xi^k)(\delta^j_l + \partial_l \xi^j) S_{ij} \tag{10}$$

$$\approx S_{kl} + S_k j \partial_l \xi^j + S_{il} \partial_k \xi^i \tag{11}$$

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3) Define the completely antisymmetric tensor, ϵ_{ABCD} such that it flips sign on any interchange of adjacent subscripts. In a coordinate system in which g_{ij} is diagonal with only 1 or -1 on the diagonal, define $\epsilon_{0123} = 1$. Argue that in a general coordinate system, $\epsilon_{0123} = \sqrt{|g|}$

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The coordinate transformation of this will be

$$\tilde{\epsilon}_{nmpq} = \frac{\partial_{x^i}}{\partial_{\tilde{x}^n}} \frac{\partial_{x^j}}{\partial_{\tilde{x}^m}} \frac{\partial_{x^k}}{\partial_{\tilde{x}^p}} \frac{\partial_{x^l}}{\partial_{\tilde{x}^q}} \epsilon_{ijkl} \tag{12}$$

Now, $\tilde{\epsilon}_{nmpq}$ and ϵ_{ijkl}, so that $ijkl$ and $nmpq$ must both be permutations of 1234 with sign being the sign of the permution (if it is an even permutation, then sign is +. and if odd, the sign is -).

Now consider the matrix $L^i_n = \frac{\partial x^i}{\tilde{x}_n}$ Consider the term with $nmpq = 1234$ every term of $\tilde{\epsilon}_{nmpq}$ will be plus or minus this. Each term of $\tilde{\epsilon}_{1234}$ takes one coefficient of the matrix L from each column of the matrix, and from each separate row, and multiplies them together, and then multiplies by the row permutation chosen and adds them. This is just the definiton of the determinant of L. So, $\tilde{e}psilon_{1234}$ will just be the determinant of L, and

$$\tilde{\epsilon}_{mnop} = det(L) \epsilon_{nmpq} \tag{13}$$

But

$$\tilde{g}_{kl} = L^i_k g_{ij} {L^T}^j_l \tag{14}$$

so

$$det(\tilde{g}_{kl}) = (detL)^2 det(g_i j) = -det(L)^2 \tag{15}$$

since $det(g_{ij}) = -1$. Thus $detL = \pm sqrt|detg_{ij}|$ or

$$\tilde{\epsilon}_{mnop} = \pm\sqrt{|detg_{ij}|}\epsilon_{nmpq} \tag{16}$$

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4.) Show that the null geodesics in the two metrics

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{17}$$

$$ds^2 = e^\phi(g_{\mu\nu}dx^\mu dx^\nu) \tag{18}$$

are the same (all of $g_{\mu\mu}$ and ϕ are functions of the coordinates. (The affine parameterisation along the geodesics will not be the same. Recall that the affine parameterisation is such that the second order geodesic equation for the null rays are the same as those for spacelike or timelike geodesics.)

$$L = \int e^\phi(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds})ds \tag{19}$$

where ds here is the afficne parameter, not the path length which is zero. Varying this gives us

$$\delta L = \int \delta e^\phi((g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds})) + \int e^\phi\delta((g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds})_d s \tag{20}$$

Since for a null curve $(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds})$ is zero, the first variation is zero. This just leaves the second, which becomes

$$\delta L = \int\left[-\frac{d}{ds}e^\phi\frac{dx^\mu}{ds}\delta^\mu_\rho + e^\phi\partial_\rho g_\mu\nu\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right]\delta x^\rho ds \tag{21}$$

From which we get the equations of motion. Now define $\lambda = \int e^{-\phi}ds$ and the equation becomes

$$e^{-\phi}\left[-\frac{d}{d\lambda}\frac{dx^\mu}{d\lambda}\delta^\mu_\rho + \partial_\rho g_\mu\nu\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] = 0 \tag{22}$$

Thus expressed in terms of λ the solutions are independent of ϕ Ie the curves are the same, expcept for the parameterization.

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5. The Flamm metric was rewriting of the spatial part of the Schwartzschild metric $\frac{1}{1-\frac{2M}{r}}dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$ by limiting the 4 dimentional metric

$$dw^2 + dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{23}$$

with the constraint that w is a function of r, ie w(r) What is w as a function of r to give the above spatial metric of the Schwartzschild metric?

Note that one can define a regular spacetime metric by

$$ds^2 = dt^2 - (d(w(r))^2 + dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)) \tag{24}$$

which is completely regular at r=2M.

Let us look at the metric

$$ds^2 = dw^2 + dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2)$$

with $w = w(r)$. The $dw = w'(r)dr$ with w' the derivative of w with respect to its argument. Thus we want

$$w'^2 dr^2 + dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2)$$

to be the spatial part of the spatial Schw. metric, or

$$w'^2 + 1 = \frac{1}{1 - \frac{2M}{r}} \tag{25}$$

$$w' = sqrt\frac{2M}{r - 2M} \tag{26}$$

This gives

$$w(r) = \int \sqrt{\frac{2M}{r - 2M}} dr = 2\sqrt{2M(r - 2M)}$$

where we have chosen the integration constant so that $w = 0$ when $r = 2M$ We can thus extend the solution to negative w by simply taking the negative square root, or

$$r = \frac{w^2}{8M} + 2M. \tag{27}$$

which is a parabola is w, r space. It is a smooth curve. Thus the Flamm metric connects two versions of flat spacetime, one for w¿¿0 and one for w¡¡0.

If we change coordinates, so that we use w instead of r as the spatial coordinate, we get

$$\frac{\frac{dr^2}{1-2M/r} = \frac{\frac{w^2}{8M}+2M}{\frac{w^2}{8M}}(wdw}{4M)^2} \tag{28}$$

$$= \frac{(w^2 + 16M^2)}{w^2} \frac{w^2}{16M^2} dw^2 = \frac{(w^2 + 16M^2)}{16M^2} dw^2 \tag{29}$$

which is completely regular even at r=2M, and

$$r^2 = (\frac{w^2}{8M} + 2M)^2 \tag{30}$$

which is also regular everywhere.

Ie, the apparent singularity at r=2M is just a coordinate singularity. It is like the singularity of the metric $ds^2 = R^2 d\theta^2$ (R is a constant) if expressed in coordinates $x = R\cos\theta), \; y = R\sin(theta)$. Then $x^2+y^2 = R^2$, or $y^2 = R^2 - x^2$ and $ds^2 = dx^2 + dy^2$ becomes

$$ds^2 = dx^2 + (\frac{dy}{dx})^2 dx^2 = (1 + \frac{x^2}{R^2 - x^2}) = \frac{R^2}{R^2 - x^2} dx^2$$

Ie, just like the Schwartzschild metric, this seems to have singularities at $x = \pm R$ rather than $r = 2M$. But that singularity is just a coordinate singularity. The singularity in the temporal part of the Schw. metric is more serious, but also turns out to be a purely coordinate singularity. It took General Relativists about 40 years to realise this in all its ramifications although there were plenty of hints there already by the 1920s.

6.a) The flat spacetime metric in polar coordinates is

$$ds_0^2 = dt^2 - dr^2 - r^2(d\theta^2 + sin(\theta)d\phi^2) \tag{31}$$

Defining $u = t - r$ or $du = dt - dr$ which gives $dt = du + dr$, we have

$$ds_0^2 = (dt + dr)^2 - dr^2 - r^2 d\Omega^2 = du^2 + 2du dr - r^2 d\Omega^2 \tag{32}$$

$$d\Omega^2 = (d\theta^2 + sin(\theta)d\phi^2) \tag{33}$$

$$\eta_{\tilde{u}\tilde{u}} = 1, \quad eta_{\tilde{u}r} = eta_{r\tilde{u}} = 1, \quad eta_{\theta\theta} = r^2, \quad eta_{\phi\phi} = r^2 sin(\theta)^2 \tag{34}$$

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c)If we have

$$ds^2 = (1 + F(r))du^2 + 2(1_G(r))du dr - r^2 d\Omega^2 \tag{35}$$

which we can always choose, for a spherically symmetric metric since the last term is just the metric on the surface of a sphere where r is taken to the "radius" of the sphere (ir rather the square root of the area of the sphere over 4π). There can be no terms like $drd\theta$ or $drd\phi$ or $dud\theta$ or $dd\phi$ since they would choose spacific directions in the $\theta, \; \phi$ directions and that would not be spherically symmetric. Furthermore, any $H(r)dr^2$ could allways be absored into a new definition of u.

Note also that any constant value of $G(r) = C + r\tilde{G}(r)$ could always be aborbed by defining $\hat{u} = du/(1 + C)$ Ie, $G(r)/r$ must be finite at $r = 0$.

Thus $h_{uu} = F(r)$ and $h_{ur} = h(ru) = G(r)$.

d)The derivatives of the metric are symbols are

$$\partial_r g_{uu} = F'(r); \quad \partial_r g_{ur} = G'(r), \tag{36}$$

$$\partial_r g_{\theta\theta} = -2r, \partial_{\phi\phi} = -2r sin(\theta^2 \tag{37}$$

$$\partial_\theta g_{\phi\phi} = 2r^2 sin(\theta)cos(\theta) \tag{38}$$

Since

$$\Gamma_{ijk} = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ik} - \partial_i g_{jk}$$

We have that the terms of

$$\Gamma_{uur} = -\frac{1}{2}F'; \quad \Gamma_{uru} = \Gamma_{ruu} = \frac{1}{2}F' \tag{39}$$

$$\Gamma_{urr} = -0; \quad \Gamma_{rru} = G'\Gamma_{r\theta\theta} = r; \quad \Gamma_{\theta r\theta} = -r; \tag{40}$$

$$Gamma_{r\theta\theta} = rsin(\theta)^2; \quad \Gamma_{\theta r\theta} = -rsin(\theta)^2 \tag{41}$$

and all the rest are 0.

Also

$$g^{ru} = g^{ur} = \frac{1}{(1+G)} \tag{42}$$

$$g^{rr} = -1 + F/(1+G)^2; \quad g^{uu} = 0 \tag{43}$$

$$g^{\theta\theta} = -\frac{1}{r^2}: \quad g^{\phi\phi} = -\frac{1}{r^2 sin(\theta)^2} \tag{44}$$

Since η_{ij} is just flat spacetime, the curvature tensor of all terms which do not contain F or G or their derivatives are zero.

If we want to lok at R_{rr}, The Riemann tensor componments must have two r indicees . Given that the contraction contains g^{ki} that we contract into $R_{....}$ cannot have an r index, the only possibilities are that this must be $g^{\phi\phi}$ or $g^{theta\theta}$. Ie

$$R_{rr} = R_{\theta r\theta r}g^{\theta\theta} + R_{\phi r\phi r}g^{\phi\phi} \tag{45}$$

Substituting into the equation for $R_{\theta r\theta r}$ and $R_{\phi r\phi r}$ we get that this term is proportional to G'. The vacuum equations thus say that G'=0. But we have argues that G has no constant term, so this means that $G(r) = 0$ and $g^{ru} = 1$

If we look at

$$R_{\theta\theta} = R_{\phi\theta\phi\theta}g^{\phi\phi} + 2R_{u\theta r\theta} = -F'r + F = 0$$

or

$$F = \frac{\tilde{C}}{r} \tag{46}$$

It will turn out that $\tilde{C} = -2M$ twhere M is the mass as determined by the period of the circular orbital time far from $r = 0.$, the metric is

$$ds^2 = (1 - \frac{2M}{r})du^2 + 2dudr - r^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{47}$$

If we define $t = u + \int \frac{1}{1-\frac{2M}{r}}dr$ we get

$$ds^2 = (1 - \frac{2M}{r}dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\Omega^2) \tag{48}$$

which is exactly the Droeste form of the Schwarzschild metric.