Physics 530-23
Assignment 2

1) Linearized Equations:
Consider the metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \beta\frac{2M}{r})(dx^2 + dy^2 + dz^2) \tag{1}$$

find the deflection of light as a funtion of β. (Keep terms only to lowest order in M)

(The theory with $\beta = 0$ was Einstein's original theory of gravity, while $\beta = -1$ corresponds to the Nordstom theory)

Hint: You could do a sucsessive approximation by finding the path of a light ray travelling in the x direction, along z=0, y=b for M=0. Find the equation for the change in the tangent vector to the real path by integrating along the above path. The angular change in direction of the tangent vector will give the deflection angle.

2)

$$\tilde{S}^i{}_j(\tilde{x}(x)) = \frac{\partial\tilde{x}^i}{\partial x^k}\frac{\partial x^l}{\partial\tilde{x}^j}S^k{}_l(x) \tag{2}$$

Show that if $\tilde{x}^i = x^i + \xi^i(x)$. Find the change in the components of S to lowest (first)order in ξ.

3) Define the completely antisymmetric tensor, ϵ_{ABCD} such that it flips sign on any interchange of adjacent subscripts. In a coordinate system in which g_{ij} is diagonal with only 1 or -1 on the diagonal, define $\epsilon_{0123} = 1$. Argue that in a general coordinate system, $\epsilon_{0123} = \sqrt{|g|}$

4.) Show that the null geodesics in the two metrics

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{3}$$

$$ds^2 = e^\phi(g_{\mu\nu}dx^\mu dx^\nu) \tag{4}$$

are the same (all of $g_{\mu\mu}$ and ϕ are functions of the coordinates. (The affine parameterisation along the geodesics will not be the same. Recall that the affine parameterisation is such that the second order geodesic equation for the null rays are the same as those for spacelike or timelike geodesics.)

5. The Flamm metric was rewriting of the spatial part of the Schwartzschild metric $\frac{1}{1-\frac{2M}{r}}dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$ by limiting the 4 dimentional metric

$$dw^2 + dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{5}$$

with the constraint that w is a function of r, ie w(r) What is w as a function of r to give the above spatial metric of the Schwartzschild metric?

Note that one can define a regular spacetime metric by

$$ds^2 = dt^2 - (d(w(r))^2 + dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)) \tag{6}$$

which is completely regular at r=2M.

6. a)Show that the metric

$$ds^2 = du^2 + 2dudr - r^2(d\theta)^2 - sin(\theta)^2 dphi^2)$$

is just a coordinate transformation of the flat metric

$$ds^2 = dt^2 - dx^2 + dy2 + dz^2$$

using the normal sperically symmetric transformation of x, y, z to r, θ, ϕ.

Thus its curvature is zero. This is called a null metric since the r coordinate axis ($\{u, \theta, \phi\}$ all consants) has zero length for its tangent vector. (Note that the metric is indepenent of u and thus the tangent vector to the u axis is a Killing vector).

b)What coordinate transformation of of t to u transforms the sperically symmetric flat metric to this metric?

c)Now consider the metric

$$ds^2 = (1 + F(r)du^2 + 2(1 + G(r))dudr - r^2(d\theta)^2 - sin(\theta)^2 dphi^2 \tag{7}$$

d) Find the inverse metric, the Christoffel symbols of the first kind (all indeces down).for this metric, and find the components of the Riemann tensor.

e) Find the component R_{uu} and show that if $R_{ij} = 0$, which was Einstein's equation for empty space then this equation implies that $G(r)$ is a constant.

f) From the $R_{rr} = 0$ equation, and assuming that we take $G(r) = 0$, show that $F(r) = \frac{C}{r}$