Physics 530-21
Assignment 3

1) Linearized Equations:
Consider the metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \beta\frac{2M}{r})(dx^2 + dy^2 + dz^2) \tag{1}$$

find the deflection of light as a funtion of β. (Keep terms only to lowest order in M)

(The theory with $\beta = 0$ was Einstein's original theory of gravity, while $\beta = -1$ corresponds to the Nordstom theory)

Hint: You could do a sucsessive approximation by finding the path of a light ray travelling in the x direction, along z=0, y=b for M=0. Find the equation for the change in the tangent vector to the real path by integrating along the above path. The angular change in direction of the tangent vector will give the deflection angle.

2)i)Consider the linearized metric

$$ds^2 = dt^2 + 2\vec{J}\frac{ydx - xdy}{r^3}dt - (dx^2 + dy^2 + dz^2) \tag{2}$$

where J is a small constant and $r^2 = x^2 + y^2 + z^2$. Show that this obeys the Einstein equations for a linerized metric such that $h_{xt} = Jy/r^3;\ \ h_{yt} = -Jx/r^3$, and this obeys

$$\partial_\nu \bar{h}^\nu{}_\mu = 0; \quad \Box \bar{h}_{\mu\nu} = 0 \tag{3}$$

ii) Change coordinates to

$$x = r\sin(\theta)cos(\phi) \tag{4}$$

$$y = r\sin(\theta)sin(\phi) \tag{5}$$

$$z = r\cos(\theta) \tag{6}$$

and show that the metric now is

$$ds^2 = dt^2 - 2\frac{J}{r}sin(\theta)^2 d\phi dt - (dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)). \tag{7}$$

Show that the proper time of particles travelling along the paths $r = r_0;\quad \Theta = \pi/2;\quad \phi = \pm\omega t$ from $\phi = 0$ to $\phi = \pm\pi$ differ.

Similarly show that particles travelling along the same paths in rotating coordinates in flat spacetime also take different times.

$$ds^2 = (1 - \Omega^2 r^2)dt^2 + 2\Omega r^2 dt d\varphi - (dr^2 + r^2(d\theta)^2 + \sin(\theta)^2 d\varphi^2 \tag{8}$$

which is flat spacetime in polar coordiantes but with $\phi = \varphi - \Omega t$.

Ie, this metric acts as if the spacetime at radius r_0 were rotating.

(J is the angular momentum of the source assumed to be pointing in the z direction)

3. Consider the metric

$$ds^2 = dt^2 - (1 + \frac{2M}{r})dr^2 - r^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{9}$$

Find the geodesic equations of this metric, Argue that $\theta = \pi/2$ should be a solution.

Find the equation for $r(\phi)$, with intial conditions at $\phi = 0$ such that $r = r_0 >> 2M$ and $\frac{dr}{d\phi} = 0$, and show that it is independent of the the velocity of the particle, or if the particle was a massive or a massless particle. Coming in from infinity and going out to infinity the particle is "deflected". What is the angle of deflection?

4. i) Consider the metric

$$ds^2 = dt^2 - dz^2 - (1 + 2h(t - z))dx^2 - (1 - 2h(t - z))dy^2 \tag{10}$$

Show that this is a solution to the linearized empty space Einstein equations for an arbitrary function h.

ii) Show that the curve $x = y = z = 0$ is a geodesic of the above metric.

iii)Consider the metric

$$ds^2 = (1 + \ddot{h}(U)\frac{(X^2 - Y^2)}{2})dU^2 + 2dUdZ - dX^2 - dY^2 \tag{11}$$

What are the components of $g_{\mu\nu}$ and $g^{\mu\nu}$? If $h = 0$ this metric is flat spacetime in coordinates $U = t - z$. What are the components of $h_{\mu\nu}$. What is the expression for $\Box$ associated with the background metric?

Show that $h_{\mu\nu}$ obeys the linearized Einstein equations and the gauge condition $\partial_\nu \bar{h}_\mu{}^\nu = 0$

$$\Gamma^X_{uu} = \ddot{h}(u)X \tag{12}$$
$$\Gamma^Y_{uu} = -\ddot{h}(u)Y \tag{13}$$
$$\Gamma^Z_{uX} = \ddot{h}(u)X \tag{14}$$
$$\Gamma^Z_{uY} = -\ddot{h}(u)Y \tag{15}$$
$$\Gamma^Z_{uu} = (\frac{d^3}{du^3}h(u))X^2)/2 - (\frac{d^3}{du^3}h(u))Y^2)/2 \tag{16}$$

Show that this metric obeys the full non-linear Einstein equations.

Ie, the linear solution is also a solution of the full equations.

Is $\{X = X_0, Y = Y_0, Z = Z_0\}$ a solution to the Geodesic equation?

Note that to at most linear order in h the two metrics are the same, just in different coordinates.