Physics 530-25
Assignment 3

1) Linearized Equations:
Consider the metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \beta\frac{2M}{r})(dx^2 + dy^2 + dz^2) \tag{1}$$

find the deflection of light as a funtion of β. (Keep terms only to lowest order in M)

(The theory with $\beta = 0$ was Einstein's original theory of gravity, while $\beta = -1$ corresponds to the Nordstom theory)

Hint: You could do a sucsessive approximation by finding the path of a light ray travelling in the x direction, along z=0, y=b for M=0. Find the equation for the change in the tangent vector to the real path by integrating along the above path. The angular change in direction of the tangent vector will give the deflection angle.

2)i)Consider the linearized metric

$$ds^2 = dt^2 + 2\vec{J}\frac{ydx - xdy}{r^3}dt - (dx^2 + dy^2 + dz^2) \tag{2}$$

where J is a small constant and $r^2 = x^2 + y^2 + z^2$. Show that this obeys the Einstein equations for a linerized metric such that $h_{xt} = Jy/r^3$; $h_{yt} = -Jx/r^3$, and this obeys

$$\partial_\nu \bar{h}^\nu{}_\mu = 0; \quad \Box \bar{h}_{\mu\nu} = 0 \tag{3}$$

For the linearized metric, we want $\partial_\rho h^\rho{}_\sigma = 0$ as a coordinate condition, and h=0. We raise the index with $\eta^{\mu\nu}$ which in this is the same matrix as $\eta_{\mu\nu}$. Thus

$$h^\rho{}_t = \sum_\rho \eta^{\rho\rho} h_{t\rho} = -h_{t\rho} = -\{0, Jy/r^3, -Jx/r^3, 0\} \tag{4}$$

so

$$\partial_\rho h_t{}^\rho = \partial_x(-Jy/r^3) + \partial_y Jx/r^3 \tag{5}$$
$$= 3Jyx/r^5 - 3Jxy/r^5 = 0 \tag{6}$$

and similarly for the othe rows where nothing depends on t.

Then the Einstein equaiton is that $\Box h_{\mu\nu} = 0$ Since nothing depends on t, this is just $-\nabla^2 h_{\mu\nu}$. Now, $h_{tx} = J\partial_y \frac{1}{r}$and thus $-\nabla^2 \partial_y \frac{1}{r} = -\partial_y \nabla^2 \frac{1}{r} = -\partial_y 0 = 0$ for everywhere except at $r = 0$ and similarly for the h_{ty} term.

Ie this is a solution to the Einstein lineaized field equaitons.
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ii) Change coordinates to

$$x = r\sin(\theta)cos(\phi) \tag{7}$$
$$y = r\sin(\theta)sin(\phi) \tag{8}$$
$$z = r\cos(\theta) \tag{9}$$

and show that the metric now is

$$ds^2 = dt^2 - 2\frac{J}{r}sin(\theta)^2 d\phi dt - (dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)). \tag{10}$$

Show that the proper time of particles travelling along the paths $r = r_0$; $\Theta = \pi/2$; $\phi = \pm\omega t$ from $\phi = 0$ to $\phi = \pm\pi$ differ.

I clearly chose theta to measure the angle from the equator not from the pole. So $sin \leftrightarrow cos$.

Clearly the transformatio now the flat metric $\eta_\mu\nu$ is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 - \cos(\theta)^2 d\phi^2) \tag{11}$$

so we just need to translate off diagonal terms.

$$\tilde{h}_{tr} = h_{tx}\frac{\partial x}{\partial r} + h_{ty}\frac{\partial y}{\partial r} \tag{12}$$
$$= J(\ cos(\theta)\ sin(\phi)\ cos(\theta)cos(\phi)\partial_r(1/r^2) - cos(\theta)\ sin(\phi)\ cos(\theta)cos(\phi)(\pm 3) \tag{13}$$
$$\tilde{h}_{t\theta} = h_{tx}\frac{\partial x}{\partial\theta} + h_{ty}\frac{\partial y}{\partial\theta} = 0 \tag{14}$$
$$\tilde{h}_{t\phi} = 0 \tag{15}$$
$$\tilde{h}_{t\phi} = Jy\partial x\partial\phi - Jxpartialy\partial\phi \tag{16}$$
$$= J/rcos(\theta)^2 sin(\phi)(-sin(\phi)) - Jcos(\theta)^2 cos(\phi cos(\phi) \tag{17}$$
$$= J/rcos(\theta)^2 \tag{18}$$

The proper time along a path with only a ϕ component of the velocity, its proper time will be

$$1 = g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^\phi - g_{\phi\phi}(u^\phi)^2 \tag{19}$$

Now $\frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \omega$ so

$$1 = (u^t)^2(1 + 2(J/r)cos(\theta)^2(\pm\omega) - r^2 cos(\theta)^2\omega^2 \tag{20}$$

or

$$(\frac{ds}{dt})^2_\pm = (u^t)^{-2} = (1 + 2(J/r)cos(\theta)^2(\pm\omega) - r^2 cos(\theta)^2\omega^2 \tag{21}$$

and

$$(\frac{ds}{dt})_+ - (\frac{ds}{dt})_- \approx \frac{2J\ cos(\theta)^2/r}{\sqrt{1 - r^2 cos(\theta)^2\omega^2}} \tag{22}$$

Ie it takes different proper times to travel the two directions around the object at r=0. It is as if the space were rotating.

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Similarly show that particles travelling along the same paths in rotating coordinates in flat spacetime also take different times.

$$ds^2 = (1 - \Omega^2 r^2)dt^2 + 2\Omega r^2 dt d\varphi - (dr^2 + r^2(d\theta)^2 + \sin(\theta)^2 d\varphi^2 \tag{23}$$

which is flat spacetime in polar coordiantes but with $\phi = \varphi - \Omega t$.

Ie, this metric acts as if the spacetime at radius r_0 were rotating.

(J is the angular momentum of the source assumed to be pointing in the z direction)

One gets the same effect in flat spacetime in rotating coordinates– the proper time to go in the two different direction is different due to the rotation

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3. Consider the metric

$$ds^2 = dt^2 - (1 + \frac{2M}{r})dr^2 - r^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{24}$$

Find the geodesic equations of this metric, Argue that $\theta = \pi/2$ should be a solution.

$$\frac{dt}{ds} = E^2 \tag{25}$$

$$\frac{d\phi}{ds} = l/r^2 \tag{26}$$

The equations must be symmetric with $\theta \to \pi - \theta$, and since we have a second order equaiton for $theta$, if $\frac{d\theta}{ds} = 0$ initially and $\theta = \pi/2$ it has nothing which will choose what dieection it would deviate. So it must stay the same ($\theta = \pi/2$ and $\frac{d\theta}{ds} = 0$.

Then we have (with α being either ± 1 or 0.

$$\frac{r/2M}{r/2M - 1}\frac{dr}{ds^2} = E^2 - \alpha - \frac{l^2}{r^2} \tag{27}$$

$$(\frac{dr}{d\phi})^2(\frac{d\phi}{ds^2})^2 = ((E^2 - \alpha)/l^2 - \frac{1}{r^2})/(1 + 2M/r) \tag{28}$$

or

$$(\frac{d(\mu)}{d\phi})^2 = \frac{1}{(1 + 2M\mu)}(\frac{E^2 - \alpha}{l^2}) - \mu^2) \tag{29}$$

If initally $r = r0 >> 2M$ and $\frac{dr}{d\phi} = 0$ we have

$$0 = (\frac{E^2 - 1}{l^2} - \mu_0^2)/(1 + 2M\mu) \tag{30}$$

$$\frac{E^2 - 1}{l^2} = \mu_0^2 \tag{31}$$

and thus

$$(\frac{d(\mu)}{d\phi})^2 = (u_0^2 - \mu^2)/(1 + 2M\mu) \tag{32}$$

Note that this independent of α, E, or l and depends only on μ_0 the equation is equivalent no matter whether the particle were light, tachyon, or orinary matter. Ie, the motion would be the same for all.

Since the equations are the same, and since we assume that $u_0 << 1/2M$, we can do successive approx.

First solve with M=0. Then the solution is

$$\mu = \mu_0 cos(\phi)$$

Nowplug This into the $M\mu$ term, so we have

$$\frac{1 + 2M\mu_0 \cos(\phi)(d\mu}{d\phi})^2 = (\mu_0^2 - \mu^2) \tag{33}$$

$$\mu \approx \mu_0 \ \cos(\int (1 + M\mu_0 \ \cos(\phi))d\phi) = \mu_0 \ cos(\phi - Mu_0 sin(\phi)) \tag{34}$$

At$\phi \approx -\pi/2$ and $\phi \approx \pi/2$ we get u=0, so $\phi - M\mu_0 \ \sin(-\pi/2) = \pi/2$ and $\phi - M\mu_0 \sin(\pi/2) = \pi/2$ we get that the total deflection is $2Mu_0$.

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Find the equation for $r(\phi)$, with intial conditions at $\phi = 0$ such that $r = r_0 >> 2M$ and $\frac{dr}{d\phi} = 0$, and show that it is independent of the the velocity of the particle, or if the particle was a massive or a massless particle. Coming in from infinity and going out to infinity the particle is "deflected". What is the angle of deflection?

4. i) Consider the metric

$$ds^2 = dt^2 - dz^2 - (1 + 2h(t - z))dx^2 - (1 - 2h(t - z))dy^2 \tag{35}$$

Show that this is a solution to the linearized empty space Einstein equations for an arbitrary function h.

——————————————— The only non-zero components of $h_{\mu\nu}$ are h_{xx}, h_{yy} and they only depend on t and z, So $\partial_\rho h^\rho_m u$ has to be zero, since the one values for ρ are x or y. Also, $\Box h_{\mu\nu}$ only has t and z derivatives, and is $(\partial_t^2 - \partial_x^2)h_{xx} = 0$. ==============================

ii) Show that the curve $x = y = z = 0$ is a geodesic of the above metric.

The equations of motion for

$$\frac{d}{ds}(1+h_{xx})\frac{dx}{ds}=0 \tag{36}$$

$$\frac{d}{ds}(1+h_{yy})\frac{dy}{ds}=0 \tag{37}$$

$$\frac{d^2z}{ds^2}-2h'(\frac{dx}{ds})^2+2h'(\frac{dy}{ds})^2=0 \tag{38}$$

From the first one $(1+h_{xx})\frac{dx}{ds}$ is constant, and since initially it is 0, it is always 0. Similarly for y. For z since $(\frac{dx}{ds})=(\frac{dy}{ds})=0$, we have $\frac{d^2z}{ds^2}=0$ and thus $dz/ds=0$ if it is so at first. ==================================

iii)Consider the metric

$$ds^2=(1+\ddot{h}(U)\frac{(X^2-Y^2)}{2})dU^2+2dUdZ-dX^2-dY^2 \tag{39}$$

What are the components of $g_{\mu\nu}$ and $g^{\mu\nu}$?

$$g_{uu}=(1+\ddot{h}(U)\frac{(X^2-Y^2)}{2}) \tag{40}$$

$$g_{uz}=g_{zu}=1 \tag{41}$$

$$g_{xx}=g_{yy}=-1 \tag{42}$$

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If $h=0$ this metric is flat spacetime in coordinates $U=t-z$. What are the components of $h_{\mu\nu}$.

$$h_{uu}=\ddot{h}(U)\frac{(X^2-Y^2)}{2}$$

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What is the expression for $\Box$ associated with the background metric?

$$\eta_{uu}=1, \quad \eta_{uz}=1, \quad eta_{xx}=eta_{yy}=-1$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Then the inverse has

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and the determinant is -1 so

$$\Box\phi = (2\partial_u\partial_z + \partial_z^2 - \partial_x^2 - \partial_y^2)\phi \tag{43}$$

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Show that $h_{\mu\nu}$ obeys the linearized Einstein equations and the gauge condition $\partial_\nu \bar{h}_\mu{}^\nu = 0$

The

$$\Gamma^X_{uu} = \ddot{h}(u)X \tag{44}$$
$$\Gamma^Y_{uu} = -\ddot{h}(u)Y \tag{45}$$
$$\Gamma^Z_{uX} = \ddot{h}(u)X \tag{46}$$
$$\Gamma^Z_{uY} = -\ddot{h}(u)Y \tag{47}$$
$$\Gamma^Z_{uu} = (\frac{d^3}{du^3}h(u))X^2)/2 - (\frac{d^3}{du^3}h(u))Y^2)/2 \tag{48}$$

Show that this metric obeys the full non-linear Einstein equations.

The $\Gamma\Gamma$ terms have contractions of the upper index of one Gamma with the lower index of the other. But there are only upper indices of X,Y and Z, and lower of X Y and u.Thus the only terms that can survive are $\Gamma^Y_{uu}\Gamma^Z_{uY}$ (and similrly for Z) But this would leave three free indices at least with value u. But the curvature is antisymmetric on at least two of its indices. and it cannot be antisymmetric with three u indices. So all of the $\Gamma\Gamma$ terms are 0.

This then leaves just the linear curvature equations. (Ie, $\partial\partial g$ type terms, which are the linearized equation.

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Ie, the linear solution is also a solution of the full equations.

Is $\{X = X_0, Y = Y_0, Z = Z_0\}$ a solution to the Geodesic equation?

No. For example the equation for X is

$$2\frac{d^2X}{ds^2} + 2\ddot{h}X(\frac{du}{ds})^2 = 0 \tag{49}$$

So only if $\frac{du}{ds}$ is 0 could the first term be 0. But we would have only u a function of s and so its derivative could not be 0

$$(1 + \ddot{h}(u)(X^2 - Y^2)/2(\frac{du}{ds})^2 = 1$$

so $\frac{du}{ds}$ is not 0
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Note that to at most linear order in h the two metrics are the same, just in different coordinates.