General Relativitiy
Bogoliubov and Annihilation operators

Linear norms

Consider a linear Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\sum_{ij}(m_{ij}p_ip_j + n_{ij}q_iq_j + r_{ij}(p_iq_j + p_jq_i)) \tag{1}$$

with all the matrices being symmetric, and real, and the usual equations of motion

$$\dot{p}_i = \frac{-\partial H}{\partial q_i} = -\sum_j(n_{ij}q_j + r_{ij}p_j) \tag{2}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \sum_j((m_{ij}p_j + r_{ij}q_j) \tag{3}$$

Let q designate a solution for all $q_i,\ p_i$ of these equations. Then define an inner product between two solutions $\tilde{q}$ and q by

$$< \tilde{q}, q >= i\sum_i(\tilde{p}_i^* q_i - p_i\tilde{q}_i^*) \tag{4}$$

The key feature of this norm is that it is conserved in time for solutions of the equations of motion. Note that this norm does not depend on any of the features of the Hamiltonian, except that it is be a linear Hamiltonian.

$$\begin{aligned}
\partial_t < \tilde{q}, q > &= i\sum_i(\dot{\tilde{p}}_i^* q_i + \tilde{p}_i^*\dot{q}_i - \dot{p}_i\tilde{q}_i^* - p_i\dot{\tilde{q}}_i^*) && (5)\\
&= i\sum_{ij}(-(n_{ij}\tilde{q}_j^* + r_{ij}\tilde{p}_j^*)q_i + \tilde{p}_i^*(m_{ij}p_j + r_{ij}q_j) && (6)\\
&\qquad -(-(n_{ij}q_j + r_{ij}p_j)\tilde{q}_i^* + p_i(m_{ij}\tilde{p}_j + r_{ij}\tilde{q}_j) && (7)\\
&= i\sum_{ij}((-n_{ij} + n_{ji})q_i\tilde{q}_j^* + (m_{ij} - m_{ji})\tilde{p}_i^* p_j && (8)\\
&\qquad +(r_{ij} - r_{ji})(q_i\tilde{p}_j^* + q_j\tilde{p}_i^* - \tilde{q}_i^* p_j + p_i\tilde{a}_j^*) && (9)\\
&= 0 && (10)
\end{aligned}$$

Ie, this norm is preserved in time for an arbitrary set of solutions. This is true even if the Hamiltonian matrices are time dependent.

If we look at $< q, q >$ we note that $< q, q >^* = < q, q >$. Ie, this is a real norm even for complex solutions. Furthermore if q is real (ie, all of the q_i, p_i are real functions), then $< q, q >=0$. Finally, since the matriceess**m, n, r** are real, if q is a solutions, then so is q^*, and

$$< q^*, q^* >= - < q, q > \tag{11}$$

Finally,

$$< q^*, q >= 0 \tag{12}$$

This is an indefinite metric on the space of solutions, like the length of vectors in Special relativity. However here for every positive norm complex solution, there exists a negative norm solution.

0.1 Quantization

We quantize this system in the usual way by defining sets of operators $\{Q_i, P_i\}$ which obey the commutation relations

$$[Q_i, Q_j] = [P_i, P_j] = 0 \tag{13}$$

$$[Q_i, P_j] = i\delta_{ij} \tag{14}$$

Furthermore, in the Heisenberg representation these operators obey exactly the same equations as the classical equations, and let us assume that we have the solution to these operator equations.

Now choose an arbitrary classical solution q, We can without loss of generality assume that its norm is $+1$. (If the norm for the first chosen q is negative, instead choose q^*, and if the norm is not 1, divide each element of q by the square root of that norm.)

Similarly the solution q^* will have norm of -1. Now define the operators

$$A^\dagger =< q, Q >= i \sum_i (p_i Q_i - q_i P_i) \tag{15}$$

$$A =< q^*, Q >= i \sum_i (p_i^* Q_i - q_i^* P_i) \tag{16}$$

since P_i, Q_i are all self adjoint Hermitian operators. Then

$$[A, A^\dagger] = \sum_{ij}[(p_i^* Q_i - q_i P_i), (p_i Q_i - q_i P_i)] \tag{17}$$

$$= -\sum_{ij}(-p_i^* q_j [Q_i, P_j] - q_i^* p_j [P_i, Q_j]) =< q, q >= 1 \tag{18}$$

Ie, these two operators have exactly the commutation relation of Annihilation and creation operators.

Now let us choose a set of such operators q_μ such that q_μ all have unit norm, and are all orthogonal to each other

$$< q_\mu, q_\nu >= \delta_{\mu\nu} \tag{19}$$

and also that also such that $< q_\mu^*, q_\nu >= 0$. One can always do this by some form of the Schmidt orthogalisation procedure. (Pick the first positive norm q_μ and its complex conjugate. Now, choose the subspace orthogonal to these two vectors. In that subspace, choose another positive norm solution and its complex conjugate. These are orthogonal to the first pair. Continue this process until one has a complete set of solutions)

Define

$$A_\mu =< q_\mu, Q > \tag{20}$$

$$A_\mu^\dagger =< q_\mu^*, Q > \tag{21}$$

The commutators will be given by

$$[A_\mu, A\nu] =< q_\mu, q_\nu^* > \tag{22}$$

which is by construction equal to zero for $\mu \neq \nu$ and by explicit calculation is zero for $\mu = \nu$.

Similarly

$$[A_\mu, A_\nu^\dagger] =< q_\mu, q_\nu > \tag{23}$$

which again by construction is zero for $\mu \neq \nu$ and is 1 for $\mu = \nu$. Ie, we have a whole set of annihilation operators, and this set is independent of the Hamiltonian, and depends only on the choice of q_i, p_i at some instant in time.

Now, consider the operator $\mathcal{H} = \sum_\mu \frac{\mu}{2}(A_\mu A_\mu^\dagger + A_\mu^\dagger A_\mu)$. This operator is a positive definite operator (

$$\langle\psi|\mathcal{H}|\psi\rangle = \sum_\mu \mu((A_\mu|\psi\rangle)^\dagger A_\mu|\psi\rangle + (A_\mu \dagger |\psi\rangle)^\dagger A_\mu^\dagger|\psi\rangle) \tag{24}$$

Each of the terms in this expression is positive for any $|\psi\rangle$ so the operators $\mathcal{H}$ is a positive definite operator.

Again

$$[A_\nu, \mathcal{H}] = \mu A_\mu \tag{25}$$

and thus if $|K\rangle$ is an eigenstate of $\mathcal{H}$ with eigenvalue K then $A_\mu|K\rangle$ is an eigenstate with eigenvalue $K - \mu$. Again, there must be a maximum value for the number of A_μ that can be applied, so that the eigenvalue for H not go negative. Applying and extra A_μ must therefor give the zero vector. This is true for all values of μ and thus one must have a state $|0\rangle$ which is annihilated by all A_μ.

Note that this state has no real physical significance, since the operator $\mathcal{H}$ was arbitrarily defined in terms of the arbitrarily defined A_μ. However it does show that for any such definition of the set of solutions q_μ, there exists a special state which has been called the "vacuum" state for this set of solutions to the classical equation. One also can call the state $A_\mu^\dagger|0\rangle$ as the state with a single "particle" in the mode q_μ.

Now consider two such sets of modes, $\{q_\mu\}$ and $\{\tilde{q}_\mu\}$. We have the two matrices

$$\alpha^*_{\mu\nu} =< q_\mu, \tilde{q}_\nu > \tag{26}$$

$$\beta^*_{\mu\nu} = - < q_\mu, \tilde{q}^*_\nu > \tag{27}$$

These matrices are called the Bogoliubov coefficients.

We can write

$$\tilde{q}_\mu = \sum_\nu \alpha^*_{\mu\nu} q_\nu + \beta^*_{\mu\nu} q^*_\nu \tag{28}$$

since

$$\alpha_\rho \nu =< q_\rho, \tilde{q}_\nu >= \sum_\nu < q_\rho, \alpha^*_{\mu\nu} q_\nu + \beta^*_{\mu\nu} q^*_\nu > \tag{29}$$

$$= \sum_\nu \alpha^*_{\mu\nu} < q_\rho, q_\nu > + \beta^*_{\mu\nu} < q_\rho, q^*_\nu >= \sum_\nu \alpha^*_{\mu\nu} \delta_{\rho,\nu} + \beta^*_{\mu\nu} 0 = \alpha^*_{\mu\nu} \tag{30}$$

as required. And similarly for β recalling that $< q^*_\rho, q^*_\nu >= -\delta_{\rho\nu}$.

Thus

$$\tilde{A}_\nu =< \tilde{q}_\nu, Q >= \sum_\mu \alpha_{\mu,\nu} < q_\mu, Q > +\beta_{\mu\nu} < q^*_\mu, Q >= \sum_\mu \alpha_{\mu\nu} + \beta_{\mu\nu} A^\dagger_\mu \quad (31)$$

Looking at the commutation relations between $\tilde{A}, \tilde{A}^\dagger$ we have

$$0 = [\tilde{A}_\nu, \tilde{A}_\rho] = \sum_\mu \alpha_{\mu\nu}\beta_\mu\rho - \alpha_{\mu\rho}\beta_\mu\nu \quad (32)$$

$$\delta_{\nu\rho} = [A_\nu, A^\dagger_\rho] = \alpha_{\mu\nu}\alpha^*_{\mu\rho} - \beta_{\mu\nu}\beta^*_{\mu\rho} \quad (33)$$

as conditions on the α and β metrices. These are the Bugoliubov relations.

Let us take the simplest case where we have only one degree of freedom. Then there is only one pair of annihilation and creation operators but they depend on the solutions which one uses to create them. Then

$$\tilde{A} = \alpha A - \beta A^\dagger \quad (34)$$

Consider the vacuum state defined by the A operator$|0\rangle$. The vacuum for the $\tilde{A}$ operator. $|\tilde{0}\rangle$ cam be written as some operator on the $|0\rangle$ which can always be written in the form $|\tilde{0}\rangle = f(A^\dagger)|0\rangle$ and the defining equation becomes

$$0 = \tilde{A}|\tilde{0}\rangle = (\alpha A + \beta A^\dagger) f(A\dagger)|0\rangle \quad (35)$$

$$= \alpha[A, f(A^\dagger)] + \beta A^\dagger f(A^\dagger))|0\rangle \quad (36)$$

But $[A, f(A^\dagger)] = \partial_{A^\dagger} f(A^\dagger)$. (Eg expand f in a Taylor seiries and note that

$$[A, {A^\dagger}^n] = \sum_r {A^\dagger}^r [A, A^\dagger] {A^\dagger}^{(n-r-1)} = n{A^\dagger}^{(n-1)} = \partial_{A^\dagger} {A^\dagger}^n \quad (37)$$

We thus have

$$\partial_{A^\dagger} f(A^\dagger) + \left(\frac{\beta}{\alpha}\right) A^\dagger f(A^\dagger) \quad (38)$$

which has solution

$$f(A^\dagger) = \mathcal{N} e^{-\frac{\beta}{2\alpha}{A^\dagger}^2} \quad (39)$$

where $\mathcal{N}$ is a normalization factor. ie, in terms of the A operators the "vacuum" $|\tilde{A}\rangle$ is a sum of pairs of particles. This state is called a squeezed state.

We want $\langle\tilde{0}||\tilde{0}\rangle = 1$ so

$$\begin{aligned}
1 &= \mathcal{N}^2\langle 0|e^{(\frac{\beta}{2\alpha})^2 A^2} e^{(\frac{\beta}{2\alpha})^{*2} A^{\dagger 2}}|0\rangle && (40)\\
&= \mathcal{N}^2\langle 0|\sum_n (\frac{\beta}{2\alpha})^n \frac{A^{2n}}{n!} \sum_m (\frac{\beta}{2\alpha})^{*m} \frac{A^{\dagger 2m}}{m!}|0\rangle && (41)\\
&= \mathcal{N}^2 \sum_n |\frac{\beta}{2\alpha}|^{2n} \frac{1}{n!^2}\langle 0|A^{2n} A^{\dagger 2n}|0\rangle && (42)
\end{aligned}$$

since $\langle 0|A^r A^{\dagger s}|0\rangle$ is zero unless $r = s$. Also since $[A,[A,....[A,A^{\dagger r}]...]] = \partial^r_{A^\dagger} A^{\dagger r} = r!$ and $A|0\rangle = 0$ we have

$$1 = \mathcal{N}^2 \sum_n \frac{(2n)!}{2^{2n} n!^2} |\frac{\beta}{\alpha}|^2 n \tag{43}$$

The Bogoliubov relations then tell us that

$$1 = |\alpha|^2 - |\beta|^2 \tag{44}$$

and thus $|\frac{\beta}{\alpha}|$ is always less than 1.

The expectation of "particle number" is

$$\langle\tilde{0}|A^\dagger A|\tilde{0}\rangle = (A|\tilde{0}\rangle)^\dagger A|\tilde{0}\rangle = |\frac{\beta}{\alpha}|^2 (A|\tilde{0}\rangle)^\dagger A|\tilde{0}\rangle = |\frac{\beta}{\alpha}|^{\langle}\tilde{0}|AA^\dagger|\tilde{0}\rangle \tag{45}$$

or

$$\langle\tilde{0}|A^\dagger A|\tilde{0}\rangle = |\beta|^2 \tag{46}$$

Ie, the particle number is just given by $|\beta|^2$. Similarly

$$\langle\tilde{0}|AA^\dagger|\tilde{0}\rangle = 1 + |\beta|^2 \tag{47}$$

Also,

$$\begin{aligned}
\langle\tilde{0}|A^2|\tilde{0}\rangle = -\frac{\beta}{\alpha}\langle\tilde{0}|AA^\dagger|\tilde{0}\rangle &= -\frac{\beta}{\alpha}(1 + |\beta|^2) && (48)\\
\langle\tilde{0}|A^{\dagger 2}|\tilde{0}\rangle &= -\frac{\beta^*}{\alpha^*}(1 + |\beta|^2) && (49)
\end{aligned}$$

One can also have a two mode squeezed state.

$$\tilde{A}_1 = \alpha_1 A_1 + \beta_1 A_2^\dagger \tag{50}$$
$$\tilde{A}_2 = \alpha_2 A_2 + \beta_2 A_1^\dagger \tag{51}$$

The commutator gives

$$0 = [\tilde{A}_1, \tilde{A}_2] = \alpha_1\beta_2 - \alpha_2\beta_1 \tag{52}$$
$$0 = [\tilde{A}_1, \tilde{A}_2^\dagger] = 0 \tag{53}$$
$$1 = [\tilde{A}_1, \tilde{A}_1^\dagger] = |\alpha_1|^2 - |\beta_1|^2 \tag{54}$$
$$1 = [\tilde{A}_2, \tilde{A}_2^\dagger] = |\alpha_2|^2 - |\beta_2|^2 \tag{55}$$

From the first third and fourth,

$$|\beta_1| = |\beta_2| \tag{56}$$
$$|\alpha_1| = |\alpha_2| \tag{57}$$

and the relative phase between β_1 and α_1 must be the same as between β_2 and α_2.

Let us assume that all are real, again for simplicity. One can make them real by altering the phases of q_1 and q_2.

The vacuum state for the ˜ modes is given by

$$\tilde{A}_1|\tilde{0}\rangle = \tilde{A}_2|\tilde{0}\rangle = 0 \tag{58}$$

or

$$A_1 + \frac{\beta}{\alpha} A_2^\dagger f(A_1^\dagger, A_2^\dagger)|0\rangle = 0 \tag{59}$$
$$A_2 + \frac{\beta}{\alpha} A_1^\dagger f(A_1^\dagger, A_2^\dagger)|0\rangle = 0 \tag{60}$$

Again, we can regard A_1 as $\partial_{A_1^\dagger}$ and A_2 as $\partial_{A_2^\dagger}$ and get two first order PDEs to solve. The solution is

$$f(A_1^\dagger, A_2^\dagger) = \mathcal{N} e^{-\frac{\alpha}{\beta} A_1^\dagger A_2^\dagger} \tag{61}$$

In this case, $A_1^\dagger$ is always accompanied by and $A_2^\dagger$ and vice versa. Ie, the "particles" always come in pairs.

In fact a mixture of single mode and two mode squeezed state is the generic situation. The general state is a product of two mode squeezed states.

Another point is that two mode squeezed state can be written as a product of single mode squeezed states.

$$e^{-\frac{\beta}{\alpha}A_1^\dagger A_2^\dagger} = e^{-\frac{\beta}{2\alpha}(\frac{A_1^\dagger + A_2^\dagger}{\sqrt{2}})^2 - (\frac{A_1^\dagger - A_2^\dagger}{\sqrt{2}})^2} \tag{62}$$

$$= e^{-\frac{\beta}{2\alpha}(\frac{A_1^\dagger + A_2^\dagger}{\sqrt{2}})^2} e^{\frac{\beta}{2\alpha}(\frac{A_1^\dagger - A_2^\dagger}{\sqrt{2}})^2} \tag{63}$$

where $B_\pm \frac{A_1 \pm A_2}{\sqrt{2}}$ are also annihilation operators for a different pair of modes. Thus the most general state is a product of single modesqueezed states. Of course if you are interested in one of the A modes for some reason, the fact that it can be written in terms of the B modes is irrelevant.

0.2 Hamiltonian diagonalisation

The above has all been about defining creation and annihilation operators in terms of a set of positive norm modes, arbitrary sets of such modes. However, sometimes one is interested in relating the annihilation operators to something else, like the energy. One can define a set of modes by

$$\partial_t p_i = i\omega p_i \tag{64}$$

$$\partial_t q_i = i\omega q_i \tag{65}$$

for all i at some time t. One thus has

$$\frac{\partial H}{\partial q_i} = -i\omega p_i \tag{66}$$

$$\frac{\partial H}{\partial p_i} = i\omega q_i \tag{67}$$

which is an eigenvalue equations for the operator

$$\mathbf{H} = \begin{pmatrix} \mathbf{r} & \mathbf{m} \\ -\mathbf{n} & -\mathbf{r} \end{pmatrix} \tag{68}$$

where the matrices $\mathbf{m}$, $\mathbf{n}$, $\mathbf{r}$ are symmetric matrices with coefficients $m_{ij},\ n_{ij},\ r_{ij}$ and H operates on the vector $\begin{pmatrix} bfq, bfp \end{pmatrix}$.

If $\mathbf{H}$ is time independent, then these solutions evolve as $e^{i\omega t}$, and these eigenmodes evolve into each other. However if $\mathbf{H}$ is time dependent, then the modes for a fixed ω at different times do not evolve into each other.

It is important to realise that the above only occures for a subset of the possible Hamiltonians. Ie, the solutions for ω may be imaginary or complex. In that Hamiltonian diagonalisation does not work.

The ω always come in $\pm$ pairs. Defining

$$S = \begin{pmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{pmatrix} \tag{69}$$

where $\mathbf{I_2}$ is the identity matrix of dimension n of the number of degrees of freedom ($\mathbf{I}$ is a $2n$ dimensional identity matrix). The eigenvalue equation is

$$0 = det(\mathbf{H} - \lambda I) = det(\mathbf{S}(\mathbf{H} - \lambda \mathbf{I})S) = det(\mathbf{H}^T + \lambda \mathbf{I}) = det(\mathbf{H} + \lambda \mathbf{I}) \tag{70}$$

because $\mathbf{m}$,$\mathbf{n}$, $\mathbf{r}$ are all symmetric matrices.

Thus if $\lambda (= i\omega)$ is an eigenvalue, so is $-\lambda$. While clearly true if ω is real, it is also true for complex or imaginary ω.

If we have two solutions with arbitrary eigenvalues $\lambda_1,\ \lambda_2$ then

$$\partial_t < q_{\lambda_1}, q_{\lambda_2} >= (\lambda_1^* + \lambda_2) < q_{\lambda_1}, q_{\lambda_2} > \tag{71}$$

but since this is zero, we find that $< q_{\lambda_1}, q_{\lambda_2} >$ can be non-zero only if $(\lambda_1^* + \lambda_2) = 0$. This is clearly true if $\lambda_1 = \lambda_2 = i\omega$ for real ω. Ie, the modes for ω real are normalizable, and orthogonal to each other. For λ real, the modes q_λ have zero norm, but have a non-zero inner product with $q_{-\lambda}$ and thus $q_\lambda + iq_\lambda$ and $q_\lambda - iq_\lambda$ are orthogonal to each other and are normalizable (with opposite signed norms).

If λ is complex, then there are four modes with eigenvalues $\lambda,\ \lambda^*,\ -\lambda,\ -\lambda^*$ which mix together. Each has zero norm, but the cross product of q_λ and $q_{-\lambda^*}$ and $q_{-\lambda}$ and q_{λ^*} are non-zero. This means that the combinations $q_\lambda \pm q_{-\lambda^*}$ and $q_{-\lambda} \pm q_{\lambda^*}$ are all orthogonal to each other, and have non-zero norms. Ie, no matter what the eigenvalues of the "Hamiltonian", we can find modes depending on the eigenstates of the Hamiltonian which can be used to make creation and annihilation operators for quantization, and the state annihilated by the associated annihilation operators will either be maxima, minima, or saddle points of the energy operator.