Physics 407-07
Christoffel Symbols

We need to define a derivative of objects which are not simply scalars. In trying to define the derivative of a vector, the problem is that addition of vectors is only defined at a point. Thus what we would want to naively define as the derivative

$$\lim_{\epsilon\to 0}\frac{V^A(\lambda+\epsilon)-V^A(\lambda)}{\epsilon} \tag{1}$$

suffers from the problem that $V^A(\lambda+\epsilon)-V^A(\lambda)$ is not defined since the two vectors inhabit different points on the spacetime, and we only defined addition for vectors at the same point.

Thus to define a derivative, we somehow have to create two vectors at the same point that we can subtract. We do this with the concept of parallelism. Ie, we somehow have to define a vector at the point $\gamma(\lambda+\epsilon)$ which is "parallel" to the vector $V^A(\lambda)$. We can then subtract that vector from the vector $V^A(\lambda+\epsilon)$ and define that limit and that derivative.

One way of doing so is to simply assume that such a concept exists, and to place certain conditions on it. Thus, let us assume that we have a concept of parallelism, which obeys some conditions:

A) The derivative of scalar function is just the ordinary derivative.

B) The derivative of the sum of two vectors is the sum of the derivatives. (Ie, parallelism is such that the parallel vector to a sum of two vectors is the vector which is the sum of the two vectors parallel to the two vectors.

C) The parallel vector to a multiple of a vector is that same multiple of the parallel vector. (Ie, these two conditions imply that parallelism is a linear operation.)

By the usual limits, this also implies the Leibnitz rule for derivatives of product.

D) The length of a parallel vector is the same as the length of the vector itself. Note that is a strong requirement of parallelism. It ties parallelism to the concept of lengths.

From these we can, if we define $\frac{D}{D\lambda}$ as the symbol designating the parallel derivative, that

$$\frac{DV^A}{D\lambda}=\frac{D}{D\lambda}(V^i\frac{\partial}{\partial x^i}^A) \tag{2}$$

$$= \frac{dV^i}{d\lambda} \frac{\partial}{\partial x^i}^A + V^i \frac{D \frac{\partial}{\partial x^i}^A}{D\lambda} \tag{3}$$

$$= \frac{dV^j}{d\lambda} \frac{\partial}{\partial x^j}^A + V^i \tilde{\Gamma}^j_{(\gamma)i} \frac{\partial}{\partial x^j}^A \tag{4}$$

where for the second line we have used the linearity of the derivative and the Leibnitz rule, and for the third, the linearity and the fact that we can express any vector in terms of the tangent vectors to the coordinate axes.

Now, Using D), let us assume that the vector V^A is parallel to itself all along the curve so that the derivative of this vector is zero. We want that this vector then have the same length all along the curve.

$$0 = \frac{dV^A V^B g_{AB}}{d\lambda} = \frac{d(V^i V^j g_{ij}}{d\lambda} \tag{5}$$

$$= \frac{dV^i}{d\lambda} V^j g_{ij} + V^i \frac{dV^i}{d\lambda} g_{ij} + V^i V^j \frac{dg_{ij}}{d\lambda} \tag{6}$$

$$= (\frac{dV^i}{d\lambda} + \tilde{\Gamma}^i_{(\gamma)k} V^k) V^j + V^i (\frac{dV^j}{d\lambda} + \tilde{\Gamma}^j_{(\gamma)k} V^k) + V^i V^j (\frac{dg_{ij}}{d\lambda} - \tilde{\Gamma}^k_{(\gamma)i} g_{kj} - \tilde{\Gamma}^k_{(\gamma)j} g_{ik}) \tag{7}$$

where we have simply added and subtracted the terms in $\tilde{\gamma}$. But since by assumption

$$0 = \frac{DV^A}{D\lambda} = (\frac{dV^i}{d\lambda} + \tilde{\Gamma}^i_{(\gamma)k} V^k) \frac{\partial}{\partial x^i}^A \tag{8}$$

we must have

$$0 = (\frac{dg_{ij}}{d\lambda} - \tilde{\Gamma}^k_{(\gamma)i} g_{kj} - \tilde{\Gamma}^k_{(\gamma)j} g_{ik}) \tag{9}$$

Now,

$$\frac{dg_{ij}}{d\lambda} = \frac{dx^k}{d\lambda} \partial_k g_{ij} \tag{10}$$

which means that the first term is linear in the tangent vector to the curve. Let us therefor put in condition

E) The parallelism $\tilde{\Gamma}^i_{(\gamma)j}$ depends on the curve $\gamma(\lambda)$ only via the tangent vector and is linear in the tangent vector. This give

$$\tilde{\Gamma}^i_{(\gamma)j} = \frac{dx^k}{d\lambda}\Gamma^i_{kj} \tag{11}$$

where the Γ^i_{kj} are called the Christoffel symbols.

This leads us to the final condition

F) The Christoffel symbol is symmetric. $\Gamma^i_{kj} = \Gamma^i_{jk}$ for all indices $i,\ j,\ k$. This is clearly a simplfying assumption. There exist theories for which this is not true. The antisymmetric part of the parallelism is then called the torsion.

With these assumptions, Γ^i_{kj} can now be written in terms of the metric.

Let us define a new symbol

$$\Gamma_{ijk} = g_{il}\Gamma^l_{jk} \tag{12}$$

(Remember the summation convention, so there is a hidden summation over l) Then our parallelism equation becomes

$$0 = (\partial_k g_{ij} - \Gamma_{jki} - \Gamma_{ikj})\frac{dx^k}{d\lambda} \tag{13}$$

or since we want the parallelism to be defined for all curves and tangent vectors

$$0 = (\partial_k g_{ij} - \Gamma_{jki} - \Gamma_{ikj}) \tag{14}$$

Now, write this same equation by renaming the indices

$$0 = \partial_k g_{ij} - \Gamma_{kij} - \Gamma_{jik} \tag{15}$$

$$0 = \partial_j g_{ki} - \Gamma_{ijk} - \Gamma_{kji} \tag{16}$$

subtracting the second and third from the first, using the symmetry of Γ in the last two indices, we get

$$0 = \partial_k g_{ij} - \partial_i g_{jk} - \partial_j gik + 2\Gamma_{kij} \tag{17}$$

or

$$\Gamma_{kij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \tag{18}$$

Since $g^{lk}\Gamma_{kij} = g^{lk}(g_{km}\Gamma^m_{ij}) = (g^{lk}g_{km})\Gamma^m_{ij} = \delta^l_m\Gamma^m_{ij} = \Gamma^l_{ij}$ we have

$$\Gamma^l_{ij} = g^{lk}\frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \tag{19}$$

This is also precisely the same as one derives by defining parallelism by means of the metric a la your exercise.

Geodesic and parallelism

The geodesic equation derived from the extreme length principle, is

$$\frac{d}{d\lambda}(g_{ij}\frac{dx^j}{d\lambda}) = \frac{1}{2}\partial_i g_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} \tag{20}$$

Now the LHS can be rewritten as

$$\frac{d}{d\lambda}(g_{ij}\frac{dx^j}{d\lambda}) = \frac{dg_{ij}}{d\lambda}\frac{dx^j}{d\lambda} + g_{ij}\frac{d^2x^j}{d\lambda^2} \tag{21}$$

$$= g_{ij}\frac{d^2x^j}{d\lambda^2} + \partial_k g_{ij}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} \tag{22}$$

So we have

$$0 = g_{ij}\frac{d^2x^j}{d\lambda^2} + \partial_k g_{ij}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} - \frac{1}{2}\partial_i g_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} \tag{23}$$

$$= g_{ij}\frac{d^2x^j}{d\lambda^2} + \Gamma_{ijk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} \tag{24}$$

or, using the inverse metric,

$$0 = \frac{d^2x^l}{d\lambda^2} + \Gamma^l_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = \frac{D}{D\lambda}\frac{dx^l}{d\lambda} \tag{25}$$

This also means that we can find what the Γ are by deriving the geodesic equations via the variational principle.

$$d\tau^2 = xtdt^2 + 2dxdt \tag{26}$$

The geodesic equations are derived from

$$\delta\int(xt\frac{dt}{d\tau}^2 + 2\frac{dx}{d\tau}\frac{dt}{d\tau})d\tau \tag{27}$$

or after doing the integrations by parts

$$\int [x\frac{dt}{d\tau}^2 - 2\frac{d}{d\tau}((xt\frac{dt}{d\tau}) - 2\frac{d}{d\tau}\frac{dx}{d\tau}]\delta t + [t\frac{dt}{d\tau}^2 - 2\frac{d^2t}{d\tau^2}]\delta x \tag{28}$$

Setting $\delta t = 0$ and then the coefficient of δx we get

$$t\frac{dt}{d\tau}^2 - 2\frac{d^2t}{d\tau^2} = 0 \tag{29}$$

Now setting $\delta x = 0$ and then the coefficient of δt we get

$$x\frac{dt}{d\tau}^2 - 2\frac{d}{d\tau}((xt\frac{dt}{d\tau}) - 2\frac{d}{d\tau}\frac{dx}{d\tau} = 0 \tag{30}$$

or

$$\frac{d^2t}{d\tau^2} - \frac{t}{2}\frac{dt}{d\tau}^2 = 0 \tag{31}$$

$$xt\frac{d^2t}{d\tau^2} + \frac{d^2x}{d\tau^2} + \frac{1}{2}x\frac{dt}{d\tau}^2 + t\frac{dt}{d\tau}\frac{dx}{d\tau} = 0 \tag{32}$$

These correspond to the equations

$$g_{ij}\frac{d^2x^j}{d\lambda^2} + \Gamma_{ijk}\frac{dx^j}{d\lambda}\frac{dx^j}{d\lambda} \tag{33}$$

and we can thus read off the Christoffel symbols of the second kind. Choosing $i = x$ which comes from the δt non-zero equation, we see

$$\Gamma_{xtt} = -\frac{t}{2} \tag{34}$$

$$\Gamma_{xtx} = \Gamma_{xxt} = \Gamma_{xxx} = 0 \tag{35}$$

Choosing $i = t$ we have

$$\Gamma_{ttt} = \frac{1}{2}x \tag{36}$$

$$\Gamma_{txt} = \Gamma_{ttx} = t \tag{37}$$

$$\Gamma_{txx} = 0 \tag{38}$$

Now we can either determine the Christoffel symbols of the first kind by using the inverse metric, or rewrite the equations so only one second derivative of a coordinate is in each equation by appropriate addition and subtraction of the equations.

The inverse metric is

$$g^{tt} = 0 \tag{39}$$

$$g^{tx} = g^{xt} = 1 \tag{40}$$

$$g^{xx} = -xt \tag{41}$$

Thus

$$\Gamma^t_{tt} = g^{tt}\Gamma_{ttt} + g^{tx}\Gamma_{xtt} = -\frac{t}{2} \tag{42}$$

$$\Gamma^x_{tt} = g^{xt}\Gamma_{ttt} + g^{xx}\Gamma_{xtt} = \frac{x}{2} + (-xt)(-\frac{t}{2} = \frac{x}{2}(1+t^2) \tag{43}$$

etc.