General Relativitiy
Curvature

Curvature

Consider two families of curves filling space, such that each set are derived by Lie dragging one set by means of the other $\gamma(\lambda)$ and $\tilde{\gamma}(\mu)$. This means that the Lie derivative of one set of tangent vectors with respect to the other is zero.

$$\pounds_{\frac{\partial}{\partial\gamma}} \frac{\partial^A}{\partial\tilde{\gamma}} = 0 \tag{1}$$

Now consider

$$\begin{aligned} D_\lambda D_\mu V^A - D_\mu D_\lambda V^A &= \lim_{\mu,\lambda\to 0} \frac{1}{\mu\lambda}[P_\lambda(P_\mu V^A(\mu,\lambda) - V^A(0,\lambda)) - (P_\mu V^A(\mu,0) - V^A(0,0)) \\ &\quad -P_\mu(P_\lambda V^A(\mu\lambda) - V(\mu,0)) - P_\lambda V^A(0,\lambda) - V^A(0,0))] \\ &= \lim_{\mu,\lambda\to 0} \frac{1}{\mu\lambda}((P_\lambda P_\mu V^A(\mu,\lambda) - P_\mu P_\lambda V^A(\mu,\lambda)) \end{aligned} \tag{2}$$

which is clearly linear in $V^A(0,0)$ in the limit.

Now,

$$D_\lambda D_\mu V^A - D_\mu D_\lambda V^A = \eta^C \xi^D (\nabla_C \nabla_D V^A - \nabla_D \nabla_C V^A) + \pounds_\eta \xi^D \nabla_D V^A \tag{3}$$

Since the last term is zero, we have that $(\nabla_C \nabla_D V^A - \nabla_D \nabla_C V^A)$ is linear in V^A and is thus a tensor in that argument. We can thus write this as

$$(\nabla_C \nabla_D V^A - \nabla_D \nabla_C V^A) = R^A{}_{BCD} V^B \tag{4}$$

$R^A{}_{BCD}$ is the Riemann curvature tensor.

Thus the components are

$$\begin{aligned} \nabla_i \nabla_j V^k &= \partial_i \partial_j V^k + (\partial_i \Gamma^k{}_{jl}) V^l) + \Gamma^k{}_{jl} \partial_i V^l \\ &\quad -\Gamma^l{}_{ij} \partial_l V^k + \Gamma^k{}_{il} \partial_j V^l - \Gamma^m{}_{ij} \Gamma^k{}_{ml} V^l + \Gamma^k{}_{im} \Gamma^m{}_{jl} V^l \end{aligned} \tag{5}$$

Antisymmetrizing over ij and using the symmetery of partial derivatives and the symmetry of the Γ we get that all the derivatives of V cancel, and are left with

$$R^k{}_{lij} = \partial_i \Gamma^k{}_{jl} - \partial_j \Gamma^k{}_{il} + \Gamma^k{}_{im} \Gamma^m{}_{jl} - \Gamma^k{}_{jm} \Gamma^m{}_{il} \tag{6}$$

Recalling that

$$\partial_r g_{km} = \Gamma_{krm} + \Gamma_{mrk}$$
$$\Gamma_{krm} = \frac{1}{2}\left(\partial_r g_k m + \partial_m g_{kr} - \partial_k g_{rm}\right) \tag{7}$$

Collecting terms we get

$$\begin{aligned} R_{klij} &= \frac{1}{2}\left(\partial_i\partial_l g_{jk} + \partial_j\partial_k g_{il} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik}\right) \\ &+\partial_i g_{km}\Gamma^m{}_{jl} - \partial_j g_{km} G^m_{il} \\ &+\Gamma_{kim}\Gamma^m{}_{jl} - \Gamma_{kjm}\Gamma^m{}_{il} \end{aligned} \tag{8}$$

We could reexpress the second line in terms of the Γ or in term of derivatives of the metric. In either case it would be even messier.

Meaning of curvature

That the Lie derivative of tangents to the two sets of curves are zero, means that the curves close. Ie, If on travels a distance $\delta\lambda$ along the first curve, and then $\Delta\mu$ along the second, on gets to the same point as if one travelled $\Delta\mu$ along the second set first and then $\Delta\lambda$ along the the first.

Now, consider that one made V^A to be parallel to itself along each of the curves. Ie, one made $D_\lambda V^A = 0$ along the curve starting at $\lambda = \mu = 0$. and ending at $\mu = 0, \lambda \neq 0$ The resulting vector will be parallel to $V^A(0,0)$ along γ. Then take that resultant Vector and parallel transport along the curve $\tilde{\gamma}$ to the point λ, μ from $\lambda, \mu = 0$. Now carry out the two transports in the opposite order (ie along $\tilde{\gamma}$ first and then along γ to the same final point. One gets a vector which is parallel to $V^A(0,0)$. But it is not the same as the first parallel vector. Instead the difference is proportional to $R^A{}_{BCD}V^B\eta^A\xi^D\delta\lambda\delta\mu$ for small λ and μ.

Since parallelism preserves lengths, both vectors have the same lenth, but point in different directions. Thus, curvature preserves lenghts but creates Lorentz transformations. Ie, the two vectors are Lorentz transformations of each other.

Symmetries

It will be useful in what follows to look at normal coordinates. We have a general coordinate system $\{x^i\}$. Consider a point $\{x^i_0\}$ with tangent vectors $\partial^A_i{}_0$ at that point. Let us assume that point p of interest has $\{x^i\}$ all equal to 0. Now in the immediate vicinity of the point define

$$x^i = y^i - \Gamma^i{}_{jk}(0)y^j y^k \tag{9}$$

and let the metric tensor components in the y coordinates be designated by $\tilde{g}_{lm}(y)$ where the $\tilde{}$ desinates components in the y coordinates. Then

$$\begin{aligned} g_{AB} &= \tilde{g}_{lm}(y) dy^l_A dy^m_B = g_{ij}(x(y)) dx^i_A dx^j_B \\ &= g_{ij}(x(y)) \frac{\partial x^i}{\partial y^l} dy^i_A \frac{\partial x^i}{\partial y^m} dy^m_B \\ &\text{or} \\ \tilde{g}_{lm}(y) &= g_{ij}(x(y)) \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^m} \end{aligned} \tag{10}$$

Taking the limit as all the y go to 0, we have

$$\begin{aligned} \frac{\partial x^i}{\partial y^l} &= \delta^i_l \\ \frac{\partial^2 x^i}{\partial y^n \partial y^l} &= -\Gamma_0{}^i_{nl} \end{aligned} \tag{11}$$

Then at the point p, (where all the coordinates x and y are 0) we have

$$\begin{aligned} \frac{\partial \tilde{g}_{lm}(y)}{\partial y^n} &= \frac{\partial g_{ij}(x(y))}{\partial x^k} \frac{\partial x^k}{\partial y^n} \frac{\partial x^i}{\partial y^l} \frac{\partial x^i}{\partial y^m} \\ &\quad + g_{ij}(x(y)) \frac{\partial^2 x^i}{\partial y^n \partial y^l} \frac{\partial x^i}{\partial y^m} + g_{ij}(x(y)) \frac{\partial x^i}{\partial y^l} \frac{\partial^2 x^j}{\partial y^n \partial y^m} \\ \frac{\partial \tilde{g}_{lm}(y)}{\partial y^n}(0) &= \frac{\partial g_{lm}}{\partial x^n}(0) - g_{il}(0) \Gamma^i{}_{nm}(0) - g_{im}(0) \Gamma^i{}_{nl}(0) \\ &= 0 \end{aligned} \tag{12}$$

Ie, in the y coordianates, all of the first partial derivatives of $\tilde{g}_{ij}$ are zero, and thus the Christofel symbols in this coordinate system are 0 at the point p. The y coordinates are called Riemann normal coordinates.

Note that this also shows that the the Christofel symbols are not tensors since if a tensor evaluated on its arguments in one coordinate system is zero, then it is zero in all coordinate systems.

The Riemann tensor has a number of symmetries. Firstly it is clear from the definition that

$$R^A{}_{BCD} = -R^A{}_{BDC} \tag{13}$$

Since symmetries of components are symmetries of the tensor itself, we can go into the above coordinate system where all the first derivatives of the metric (and thus all the Γs) are zero. Then

$$\begin{aligned} R_{ijkl} &= g_{im}R^m{}_{jkl} = g_{im}\partial_k\Gamma^m{}_{jl} - \partial_l\Gamma^m{}_{jk} = \partial_k(g_{im}(\Gamma^m{}_{jl} - \partial_l\Gamma^m{}_{jk}) \\ &= \partial_k(\partial_j g_{li} - \partial_i g_{lj}) + \partial_l(\partial_j g_{ki} - \partial_i g_{kj}) \end{aligned} \tag{14}$$

where I used that the derivative of the metric was zero, and defined

$$\Gamma_{ijk} = g_{im}\Gamma^m{}_{jk} = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) \tag{15}$$

This gives

$$R_{ijkl} = \frac{1}{2}(\partial_k\partial_j g_{il} + \partial_l\partial_i g_{kj} - \partial_l\partial_j g_{ik} - \partial_k\partial_i g_{lj}) \tag{16}$$

This clearly satisfies

$$\begin{aligned} R_{ijkl} &= R_{ijlk} \\ R_{ijkl} &= R_{jikl} \\ R_{ijkl} &= R_{klij} \\ R_{ijkl} + R_{iklj} + R_{iljk} &= 0 \end{aligned} \tag{17}$$

That last symmetry can also be written as

$$R_{ijkl} + R_{iklj} + R_{iljk} - (R_{ikjl} + R_{ijlk} + R_{ilkj}) = 0 \tag{18}$$

which is completely antisymmetric in the last three indices.

In 4-D spacetime, the first two index symmetries state that for every ij, there are 6 independent kl components, and similarly for ij for each kl. Then if we regard the first two and last two indices as joint 6 dimensional indices, the symmetry of interchange of these gives us (6x6-6)/2+6= 21 independent terms. The final symmetry says that there is one additional constraints, leaving us with 20 in total. (The last 3 indices must all be different and the fourth possible value must be in the first place. But with the other symmetries one can always put any of the 4 different indices into the first, i, place). Thus there are 20 total number of possible independent components.

Since $R_{ABCD}V^aW^BU^CX^D = R_{ijkl}V^iW^jU^kX^l$, if the components have some symmetry (eg $(R_{ijkl}V^iW^jU^kX^l = -R_{ijkl}V^iW^jU^lX^k$ in any coordinate system for arbitrary vectors, then so does the tensor.

Bianci Indentities

The Riemann tensor components are of the form

$$R_{ijkl} = \partial\partial g + \partial g \partial g \tag{19}$$

Then look at

$$\nabla R = \partial\partial\partial g + \partial\partial g \partial g + \Gamma R \tag{20}$$

. where I have supressed the indices.

We go to the Riemann normal coordinate system where ∂g and thus Γ are all zero at the point of interest.. Then the only terms that survive are the $\partial\partial\partial g$ terms. Let us now insert the actual indices.

$$\begin{aligned} \nabla_i(R_{jklm}) &= \partial_i\left((\partial_l\partial_j g_{km} - \partial_m\partial_j g_{jl}\right. \\ &\quad -\partial_l\partial_k g_{jm} + \partial_k\partial_m g_{jl}) \end{aligned} \tag{21}$$

Now look at

$$\begin{aligned} &\nabla_i(R_{jklm}) + \nabla_j R_{kilm} + \nabla_k R_{ijlm} \\ -&(\nabla_j(R_{iklm}) + \nabla_i R_{kjlm} + \nabla_k R_{jilm}) \end{aligned} \tag{22}$$

which is the complete antisymetric permutation of i, j, k. Expanding R in terms of the derivatives of g, one of the indices of g will be either l or m That means that two of i, j, k will be partial derivatives. But the commutators of two ordinary partial derivatives is zero. Thus this expression will be zero. This is the Bianci identity.

Since this is tensor symmetry, it will also be true for coordinates where the Γ are not zero. Thus

$$\nabla_A R_{BCDE} + \nabla_B R_{CADE} + \nabla_C R_{ABDE} = 0. \tag{23}$$

Now we can contract this expression with g^{BD} and recalling that $\nabla_A g_{BC} = 0$ to get

$$\nabla_A R_{CE} + \nabla_B R_{CA}{}^{B}{}_{E} - \nabla_C R_{AE} = 0 \tag{24}$$

Finally, contracting with g^{CE} we get

$$\nabla_A R - 2\nabla_B R^{B}{}_{A} = 0 \tag{25}$$

or

$$\nabla_B(-G^B{}_A) = 0 \tag{26}$$

where

$$\begin{aligned} R_{BD} = g^{AC} R_{ABCD} = R^A{}_{BAD} \\ R = g^{BD} RBD \\ G_{AB} = R_{AB} - \frac{1}{2} g_{AB} \end{aligned} \tag{27}$$

This tensor G_{AB} is called the Einstein tensor and it is conserved. (A tensor is conserved if the trace with respect to any of the indices of the tensor is zero.)

Note that another tensor which is useful is the completely trace free curvature.

$$\begin{aligned} C_{ABCD} &= R_{ABCD} \\ &-\frac{1}{2}(R_{AC} g_{BD} - R_{AD}\, g_{BC} - R_{BC}\, g_{AD} + R_{BD}\, g_{AC})) \\ &-\frac{1}{6} R(g_{AC}\, g_{BD} - g_{AD}\, gBC) \end{aligned} \tag{28}$$

which is trace-free. ($g^{AC} C_{ABCD} = 0$). (Recall that $\delta^i_i = 4$)

This is called the Weyl tensor, and also has the property that if $\tilde{g}_{AB} = \Omega^2 g_{AB}$, then the Weyl tensor $\tilde{C}^A_{BCD}$ for the confomally transformed metric $\tilde{g}_{AB}$ is the same as for the original tensor $C^A{}_{BCD}$ defined for g_{AB} (Note that this is true only if one of the indicees is up and the others down. The tensor with two or more raised indices is not the same for the conformally transformed metric as for the original. . Note that C_{ABCD} is zero for all dimensions less than 4.(The symmetries demands that all of the component indices must be different from each other, and that requires are lest 4 different indices) In three dimensions, R_{ABCD} can be written in terms of R_{AB} and in two dimensions both R_{ABCD} and R_{AB} can be written in terms of R and the metric alone.

Linearized curvature

Let us write in some coordinate system that

$$g_{ij} = \eta_{ij} + h_{ij} \tag{29}$$

where the η_{ij} are assumed to be constants in spacetime, and h_{ij} are assumed all to be small, so we will keep only terms to first order in the various h_{ij}.

Then

$$g^{ij} = \eta^{ij} - \eta^{ik}\eta^{jl}h_{jl} \tag{30}$$

as can be seen by

$$\delta^i_j = g^{ik}g_{kj} = \eta^{ik}\eta_{kj} + \eta^{ik}h_{kj} - \eta^{ik}h_{kl}\eta^{lm}\eta_{mj} + O(h^2) = \eta^{ik}\eta_{kj} = \delta^i_j \tag{31}$$

In the curvature, all of the terms that go like $\Gamma\Gamma$ will be second order in h since ${\Gamma^i}_{jk}$ is written in terms of derivatives of the h and thus is first order in h, and products would be second order.

Also $g_{im}\partial_k {\Gamma^m}_{jl} = \partial_k\Gamma_{ijk} + O(h^2)$ and thus the linearized curvature to lowest order in h is the same as the above curvture in Riemann normal coordinates

$$R_{ijkl} = \frac{1}{2}(\partial_k\partial_j h_{il} + \partial_l\partial_i h_{jk} - \partial_k\partial_i h_{jl} - \partial_l\partial_j h_{ik}) \tag{32}$$

The Ricci curvature is

$$R_{jk} = \frac{1}{2}\left(\partial_k\partial_j h + \Box h_{kj} - \partial_k\partial i\eta^{il}h_{lj} - \partial_j\partial_i\eta^{il}h_{lk}\right) \tag{33}$$

where $h = \eta^{ij}h_{ij}$ and $\Box = \eta^{ij}\partial_i\partial_j$

If we write $\bar{h}_{ij} = h_{ij} - \frac{1}{2}h\eta_{ij}$, then we have

$$G_{ij} = \frac{1}{2}\Box\bar{h}_{ij} - \partial_i\partial_l\eta^{lk}\bar{h}_{kj} - \partial_j\partial_l\eta^{lk}\bar{h}_{li}) \tag{34}$$

If η_{ij} is the Minkowski metric, then $\Box$ is like a wave equation, and the other two terms are divergences of tensors. Since the small metric changes if one performs coordinate transformations, this gives us hope that perhaps those divergences can be set to zero, and the then G_{ij} is just a wave equation. (This is similar to electromagmetism, where the equation for A^i, the vector potential, is of the form

$$\Box A^i - \eta^{ij}\partial_j\partial_k A^k = J^i \tag{35}$$

and the second term can be eliminated via a guage transformation.

The linearized equations for gravity were discovered by Einstein in 1915, and represent waves of metric changes which travel at the speed of light (as he published in 1916). His calculations in 1915 (bending of light and perihelion of Mercury) were done using the linearized theory.

There were arguments until the 1970's as to whether or not gravitational waves were real, or whether they could always be eliminated by coordinate transformation "Do they travel at the speed of thought" as one physicist stated. It was only in the 1970s that the argument died out in the face of both theoretical and experimental (the change in orbit of a binary pulsar which was just what General Relativity said) work.