Physics 407-07
Equations

0.1 Vectors

Tangent vector V^A to a curve $\gamma(\lambda)$ is $V^A = \partial_\gamma^A$ Cotangent vector W_B to a function $W_B = df_B$
Product

$$V^A W_A = \frac{df(\gamma(\lambda)}{d\lambda} \tag{1}$$

Coordinates: / set of D functions $x^i(p)$ such that $x^i(p)$ values uniquely specifies a point p, and such that $x_j(p) = consts; j \neq i$ is a differntiable curve.

$$x_j(\gamma(\lambda)) = \text{const}(j \neq i), \tag{2}$$

$$x_i(\gamma(\lambda) = \lambda \tag{3}$$

i^{th} coordinate axis
Components:

$$V^A = V^i \partial_{x^i}^A \tag{4}$$

$$W_B = W_j dx_B^j \tag{5}$$

Compoents are real numbers

$$V^A W_A = V^i W_i \tag{6}$$

If x^i and $\tilde{x}^i$are two coordinate systems

$$\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^j} V^j \tag{7}$$

$$\tilde{W}_j = \frac{\partial x^k}{\partial \tilde{x}^k} W_k \tag{8}$$

$$x^i(\tilde{x}^j(x^k)) = \delta^i_k x^k \tag{9}$$

$$\delta^j_l \frac{\partial x^i(\tilde{\{x^j(\{x^k\})\}})}{x^l} = \delta^i_l \tag{10}$$

$$V^i W_i = \frac{\partial x^i}{\partial \tilde{x}^j}\tilde{V}^j \frac{\partial \tilde{x}^k}{\partial x^i}\tilde{W}_k \tag{11}$$

$$= \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^k}{\partial x^i} V^j W_k = \tilde{V}^k \tilde{W}_k \tag{12}$$

$f(\{x^i\})$ is a function of the set of coordinates $\{x^i\}$ Ie, this means the set of coordinates with i going from 0-3 or 1-4 This will often be written as $f(x^i)$ where this does not mean some specific value of i, but the whole set. (ie the $\{..\}$ is not explicitly written.

$$x^i = x^i(\{\tilde{x}^j(\{x^k\}\}) \tag{13}$$

$$\delta^i_k = \partial_{x^k} x^i = \partial_{\tilde{x}^j} x^i(\{\tilde{x}^j(\{x^j\}\})\partial_{x^k}\tilde{x}^j(\{x^k\}) \tag{14}$$

by the chain rule.

$$V^A = V^i \partial^A_i = V^i \partial_i \tilde{x}^j \partial^A_{\tilde{j}} \tag{15}$$

$$V^i = \partial_{\tilde{j}} x^i \tilde{V}^j \tag{16}$$

Also

$$V^A W_A = V^i W_j \partial^A_i d^A_j = V^i W_i = \tilde{V}^j \tilde{W}_j \tag{17}$$

$$V^A W_A = \partial_{\tilde{k}} x^i \tilde{V}^k W_i \tag{18}$$

for all V^A and thus we must have

$$\tilde{W}_j = \partial_{tildex^j} x^i W_i \tag{19}$$

Similarly in a tensor, each upper index transforms like tangent vecor while each lower index transforms like cotangent vector.

Linearized transformation.

$$\tilde{x}^j = x^j + \zeta^j(\{x_k\}) \tag{20}$$

$$x^j = \tilde{x}^j - \zeta^j(\{x_k\}) = \tilde{x}^j - \zeta^j(\{\tilde{x}_k - \tilde{x}^j - \zeta^j(\{x_k\}) \approx \tilde{x}^j - \zeta^j(\{\tilde{x}_k\} \tag{21}$$

where ζ is a small function and thus the last term in the funtion is of higher order in ζ. (This also implies that the derivatives of ζ are all small).

Thus if

$$g_{ij} = \eta_{ij} + h_{ij} \tag{22}$$

where $\eta_i j$ is of order 1 while h_{ij} is small then

$$\tilde{g}_{ij} = g_{kl}\partial_{\tilde{i}}x^k\partial_{\tilde{j}}x^l \tag{23}$$

$$\approx (\eta_{kl} + h_{kl})(\delta_i^k + \partial_i\zeta^k)(\delta_j^l + \partial_j\zeta^l)) \tag{24}$$

$$\approx \eta_{ij} + h_{ij} + \partial_i\zeta^k\eta_{kl} + \partial_j\zeta^l\eta_{li} \tag{25}$$

Since $g_{ij}g^{jk} = \delta_i^k$ so

$$g^{il} = \eta^{il} - h^{il} = \eta^{il} - \eta^{ij}h^{jk}\eta^k l \tag{26}$$

Note, raising and lowering of indices for an expression contaning h of ζ are done with η.