General Relativitiy
Hole Argument)

When Einstein thought he had shown that if he used an equation for gravity which was coordinate independent, (not least because he decided that the rotating metric

$$ds^2 = (1 - \Omega^2 r^2)dt^2 - 2r^2\Omega dt d\phi - r^2 d\phi^2 - dr^2 - dz^2 \tag{1}$$

which is a coordinate transformation of the flat cylindrical metric

$$ds^2 = dt^2 - d\tilde{\phi}^2 - dr^2 - dz^2 \tag{2}$$

$$\tilde{\phi} = \phi + \Omega t) \tag{3}$$

He found that the first did not obey the equations while the second one did. (In the first one he took $g_{t\phi} = 2r^2\Omega$ instead of $g_{t\phi} = g_{\phi t} = r^2\Omega$ as it should be. He therefore concluded that a tensor equation did not work. His friend Besso wrote him and told him that there seemed to be something wrong with the calculations, but Einstein ignored him.

He then came up with another argument. Consider a spacetime which obeys coordinate covariate equations which is a solution of the equations of motion. Since the equations are coordinate covatiant equations, this implies that if we do a coordinate transformation, the transformed metric must also satisfy the equations of motion.

Now consider a spacetime with a metric which is the solution of the equations of motion. The equations of motion will be local second order differential equations and will have some sort of intial value property. Ie, if we specify g_{ij} and a suitable selections of time derivatives of g_{ij} this will determine g_{ij} at all times. Now create a closed 4-volume at some some time to the future of that initial time t_0 on which we specify those initial values. Inside that volume we now change the coordinates, but leave them alone everywhere else. This change of coordinates will change the metric inside the volume but not outside, and thus not on the initial value surface $t = t_0$. This new metric will still solve the equations of motion for the metric. Thus you now will have two solutions (or rather and infinity of solutions since there is an infinity of coordinate transformations you could have carried out inside the volume).

This would mean that the equations of motion would be indeterminate– and infinity of answers to the same initial values, which would make physics useless. Physics is founded on the principle that the past determines the future. Thus, the theory could not be coordinate covariant. He thus had two arguments, for this conclusion, which proved that the conclusion must be right.

He wandered down the swamp of the of his Entwurf theory, which was not coordinate invariant. (Not that many many people have gotten lost on that same swamp and never got out, including Hilbert, and, far more recently, Lugenov, an excellent particle theorist (and member of the Supreme Soviet, and Rector of Moscow Unriversity).

It was only in 1915 that Einstein found his way out– Physics should be independent of the coordinates, which meant that physical properties of the world should also be independent of the coordinates. Relationships (which was what equations give you) should be expressible in terms of coordinate covarient mathematical objects.

Copyright William Unruh 2025