General Relativitiy
Linearized

Linearize the equations assuming that

$$g_{ij} = \eta_{ij} + h_{ij} \tag{1}$$

where h_{ij} are all assumed to be small so that only first order terms in h_{ij} will be kept. Then

$$\delta^i_k = g^{ij} g_{jk} = g^{ij}(\eta_{ij} + h_{jk}) = (\eta^{ij} + g^{ij} - \eta^{ij})(\eta_{jk} + h_{jk}) = \delta^i_k + \eta^{ij} h_{jk} + (g^{jk} - \eta^{jk})\eta_{jk} \tag{2}$$

from which

$$g^{ij} = \eta^{ij} - \eta^{ik}\eta^{jl} h_{ij} \equiv \eta^{ij} - h^{ij} \tag{3}$$

where the rasing of the index on h is via the metric η^{ij}

Substituting into the curvature, and keeping only terms linear in h at most, and realising that the only parts of the metric that are spatially dependent are the h, all terms of the form $\Gamma\Gamma$ can be neglected as being of order h^2.

$$R_{klij} = \frac{1}{2}\left(\partial i \partial l g_{jk} + \partial_j \partial_k g_{il} - \partial_i \partial_k g_{jl} - \partial_j \partial_l g_{ik}\right) \tag{4}$$

where the $\Gamma\Gamma$ terms are again zero because it is second order in h (Note that this depends on the backgroud metric being independent of coordinates.

Then

$$R_{ijkl} = \frac{1}{2}(\partial_i(\partial_k h_{jl} + \partial_j h_{kl} - \partial_l h_{jk}) - i \leftrightarrow j \tag{5}$$

$$= \frac{1}{2}(\partial i \partial_k h_{jl} + \partial_j \partial_k h_{ik} - \partial_j \partial_k h_{il} - \partial_i \partial_l h_{jk}) \tag{6}$$

The Ricci tensor is

$$R_{ik} = g^{jl} R_{ijkl} \approx \eta^{jl} R_{ijkl} \tag{7}$$

Now define $h = \eta^{ij} h_{ik}$ and $h^i{}_j = \eta^{ik} h_{kj}$ and $\Box = \eta^{ij}\partial_i \partial_j$ to get

$$R_{ik} = \frac{1}{2}(\Box h_{ik} + \partial_i \partial_k h - \partial_i \partial_j h^j{}_k - \partial_k \partial_j h^j{}_i) \tag{8}$$

Notice that if η_{ij} is the usual Minkowski metric $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ (taking c to be unity) then $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ then the first term is just the wave equation term. The other three equations are messier.

Now, one of the problems is the coordinate problem. Under a change in coordinates from $\tilde{x}$ to x coordinates we have

$$g_{AB} = g_{ij} dx_A^i dx_B^j = \tilde{g}_{kl} d\tilde{x}_A^k d\tilde{x}_B^k \tag{9}$$

But the coordinates are just scalar fields defined on the spacetime. This

$$d\tilde{x}_A^k = \partial_i \tilde{x}^k dx_A^i \tag{10}$$

Thus

$$g_{AB} = \tilde{g}_{kl} \partial_i \tilde{x}^k \partial_j x^l dx_A^i dx_B^j \tag{11}$$

pr

$$g_{ij} = \tilde{g}_{kl} \partial_i \tilde{x}^k \partial_j \tilde{x}^l \tag{12}$$

This is the general transformation of a metric.

0.1 scalar field

Let us look at the generalisation of the Klein Gorden field equaiton

$$\partial_t^2 \phi - \partial_x^2 \phi - \partial_y^2 \phi - \partial_z^2 \phi \tag{13}$$

The first guess would be

$$\nabla_A g^{AB} \nabla_B \phi \tag{14}$$

Teh components of this would be

$$g^{AB} \nabla_B \phi \to g^{ij} \partial_j \phi \tag{15}$$

and

$$\nabla_A g^{AB} \nabla_B \phi = \partial_i (g^{ij} \partial_j \phi) + \Gamma^i_{ki} g^{kj} \partial_j \phi \tag{16}$$

Now

$$\Gamma^i_{ki} = g^{im}\frac{1}{2}(\partial_i g_{mk} + \partial_k g_{mi} - \partial_m gki) \tag{17}$$

The first and third term cancel under the interchange of m and i which are just summed over. This leaves

$$\Gamma^i_{ki} = \frac{1}{2} g^{im}\partial_k g_{im} \tag{18}$$

Now let us look at the determinant of the square matrix $\{g_{ij}\}$– the matrix whose elements are the components of the metric. We will call this determinant g. We know that in the determinant, each term g_{ij} occurs either not at all or linearly. Let us define the Minor of g_{ij}, M^{ij} as $(-1)^{i+j}$ times the determinant of the matrix with the ith row and jth column crossed out. Then the term in g multiplying g_{ij} is just M^{ij} Also the determinant is just $g = \sum_j g_{\hat{i}j} M^{\hat{i}j}$ where *hati* is some value of i which is chosen and fixed. The sum $\sum_j g_{\hat{k}j} M^{\hat{i}j}$ where $\hat{k} \neq \hat{\imath}$ is the determinant of the matrix where the $\hat{\imath}$th row was replaced by the $\hat{k}$th row. Ie, this is the determinant of a matrix where the $\hat{k}$th row and $\hat{\imath}$th row are the same. But a matrix with two rows the same has a determinant of 0. Thus

$$\sum_j g_{\hat{k}j} M^{\hat{i}j} = g\delta^{\hat{i}}_{\hat{k}} \tag{19}$$

Thus the components of the inverse metric are just $g^{ij} = \frac{1}{g} M^{ij}$.

The derivative of g is then

$$\partial_k g = \partial_k g_{ij} M^{ij} = \partial_l g_{ij} g g^{ij} \tag{20}$$

Thus $\Gamma^i_{ki} = \frac{\partial_k g}{2g} = \frac{\partial_k \sqrt{|g|}}{\sqrt{|g|}}$ and we get

$$\nabla_A g^{AB} \nabla_B \phi = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \phi \right) \tag{21}$$

Now let us choose our coordinates such that each of the coordinates satisfies $\nabla_A g^{AB} \nabla_B x^k = 0$. These are called harmonic coordinates. (note that this is certainly true of the usual t, x, y, z coordinates since the condition is a second derivative while the coordinates are linear functions of the coordinates)

Then we want

$$\partial_i \sqrt{|g|} g^{ij} \partial_j x^k = 0 \tag{22}$$

But $\partial_j x^k = \delta_j^k$ and thus this condition becomes

$$\partial_i (\sqrt{|g|} g^i j = 0 \tag{23}$$

This is called the Harmonic coordinate condition. If we write this in terms of the linearized metric, we get

$$\frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} = \frac{1}{2} g^{kl} \partial_i g_{kl} = \frac{1}{2} (\eta^{kl} \partial_i h_{kl} = \frac{1}{2} \partial_i h \tag{24}$$

since to 0th order $g = det(\eta_{ij}) = -1$. and thus the harmoic condition becomes

$$\frac{1}{2} \partial_j h \eta^{ij} - \partial_j h^{ij} \tag{25}$$

Defining $\bar{h}_{ij} = h_{ij} - \frac{1}{2} h \eta_{ij}$, this becomes

$$\partial_j \bar{h}^{ij} = 0 \tag{26}$$

Using these harmonic coordinates we get

$$R_{ij} = \frac{1}{2} (\Box h_{ij} - \partial_i \partial_k \bar{h}_j^k - \partial_j \partial_k \bar{h}_i^k) = \frac{1}{2} \Box h_{ij} \tag{27}$$

Then the Einstein tensor is

$$G_{ij} = R_{ij} - \frac{1}{2} g^{kl} R_{kl} g_{ij} = \frac{1}{2} \Box \bar{h}_{ij} \tag{28}$$

Thus, in these coordinates, the components of the linearized Einstein tensor are just half of the Klean Gordon operator on each component of the trace reversed metric perturbation.

If

$$G_{AB} = \alpha T_{AB} \tag{29}$$

then we have

$$G_{AB} = R_{AB} - \frac{1}{2} R g_{AB} \tag{30}$$

$$g^{AB} G_{AB} = R - \frac{1}{2} R g^{AB} g_{AB} = -R \tag{31}$$

since $g^{AB} g_{AM} = \delta^A_A = 4$. Thus $-R = \alpha g^{AB} T_{AB} = \alpha T$ and

$$R_{AB} = \alpha T_{AB} + \frac{1}{2} R g_{AB} = \alpha (T_{AB} - \frac{1}{2} T g_{AB}) \tag{32}$$

For a Newtonian star, the rest mass energy density far exceeds the energy flux, the momentum flux or the stresses (they are at best of order of the velocity of sound over c smaller). Thus we can approximate the Newtonian situation by assuming that only the component $T_0 0$ is non-zero. Then $T = g^{ij} T_{ij} = \eta^{ij} T_{ij} = \eta^{00} T_{00}$ and we have

$$R_{00} = \alpha (T_0 0) - \frac{1}{2} T_{00}) = \frac{1}{2} T_{00} \tag{33}$$

But in our Harmoic coordinates, $R_{00} = \frac{1}{2} \Box h_{00}$. We found previously that in the slow motion limit, if $h_{00} = 2\phi$ where here ϕ is the Newtonian potential, then the particle travelling along a geodesic will have an equation of motion for the spatial cordinates of Newton's force equations in a gravitational field. Thus we want h_{00} to be twice the Newtonian potential. Assuming the source of the gravitational potential does not move, then ϕ depends only on the spatial coordinates, and $\Box h_0 0 = -2\nabla^2 \phi$. The Newtonian potential obeys

$$\nabla^2 \phi = 4\pi G \rho \tag{34}$$

where ρ is the mass density (or the rest mass energy density).

Thus we require $\alpha = -8\pi G$. in order that the Einstein equations be approximately the Newtonian gravitational equations in the limit that ϕ is small (or reinserting c the velocity of light, ϕ / c^2 is small) and if we demand slow motion so that time derivatives can be ignored, and non-reltivistic matter, so that the energy density dominates the energy momentum tensor.

Thus we have

$$G_{AB} = -8\pi G T_{AB} \tag{35}$$

0.2 Harmonic coordinates.

The coordinates are simply scalar fields used to label the points in spacetime. One possibility is to choose the coordinates so that the coordinate functions themselves are harmonic– ie, satisfy

$$g^{AB}\nabla_A\nabla_B x^i = 0 \tag{36}$$

Writing this in coordinate form we have

$$\frac{1}{\sqrt{|g|}}\partial_i\sqrt{|g|}g^{ij}\partial_j x^k = 0 \tag{37}$$

But in the x^i coordinates, $\partial_j x^k = \delta^k_j$ so the Harmonic coordinates lead to the requirement that

$$\partial_i(\sqrt{(|g|)}g^{ik}) = 0 \tag{38}$$

This is the condition that the coordinates all satisfy the Klein Gordon equation.

Note that this does not uniquely specify the coordinates. There are of course an infinite number of solutions to the Klein Gordon equation, and thus an infinite number of coordinates which will satisfy this condition.

If we linearize the metric around the flat spacetime metrix η_{ij} such that $g_{ij} = \eta_{ij} + h_{ij}$, where only h_{ij} depends on the coordinates, then

$$\partial_i(\sqrt{|g|}g^{ik} = \partial_i(-h^{ik} - \frac{1}{2}\eta^{mn}h_{mn}\eta^{ij}) = 0 \tag{39}$$

(any term in the matrix enters into the the terms of the determinant either not at all or linearly. Define $M^{ij} = \partial_{g_{ij}}det(\mathbf{g})$ as the minor of the term g_{ij} The minor is the determinant of the matrix $\mathbf{g}$ where you cross out the ith row and jth column times $(-1)^{i+j}$. Let us fix i. Then $det(\mathbf{g}) = \sum_j g_i j M^{ij}$ (no summation convention on i). If we take $i' \neq i$, then $\sum_j g_{i'j}M^{ij}$ would be the determinant if the ith row of the matrix $\mathbf{g}$ were replaced by the i'th row. Ie, that would be a matrix with two rows, the ith and i'th the same. But the determinant of such a matrix with two rows the same is 0. This

$$\sum_j g_{kj}M^{ij} = delta^i_k det(\mathbf{g}) \tag{40}$$

or

$$M^{ij} = det(g)g^{ij} \tag{41}$$

The linearization of $det\mathbf{g}$ will thus be $h_{ij}\partial_{g_{ij}}g = h_{ij}\eta^{ij}det\eta = h$ where the summation convention applies. Ie, the linearization of g, the determinant, is just h the trace of the perturbation.

Under and infinitessimal coordinate transformation $x^i \to x^i + \zeta^i$. the first order metric changes by

$$h_{ij} \to h_{ij} + \partial_i\zeta_k + \partial_k\zeta_i \tag{42}$$

where $\zeta_i = \eta_{ij}\zeta^k$.

Thus we would require that

$$\partial_i(h^{ik} + \partial^i\zeta^k + \partial^k\zeta^i - \frac{1}{2}\partial^k(h + 2\zeta_j\zeta^j) = \partial_i(h^{ik} - \frac{1}{2}\partial^k h) + \Box\zeta^k. = 0 \tag{43}$$

Thus if the original coordinates obeyed the harmonic condition, the new coordinates will obey the lineared harmonic condition if

$$^k = 0 \tag{44}$$

Substituting into Einstein's equations we have

$$\Box\bar{h}_{ij} = -16\pi G_N T_{ij} \tag{45}$$

as the Einstein equations for the linearized metric. The solution to these equations is of course the retarded Green's function for the Klein gordon equation

$$\bar{h}_{ij}(t,x) = -\int 16\pi G_N T_{ij}(t',x')\frac{\delta(t-t'-|x-x'|)}{4\pi|x-x'|}dt'd^3x' \tag{46}$$

where $|x-x'| = \sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}$ If T_{ij} is dominated by T_{00} (ie nonrelativistic matter where the pressures and momenta are much smaller than the rest mass energy density) then only $\bar{h}_{00}$ will be non-negligible, and

$$\eta_{ij}\bar{h}^{ij} = \eta_{ij}h^{ij} - \frac{1}{2}h\eta_{ij}\eta^{ij} = -h \tag{47}$$

and

$$h_{ij} = \bar{h}_{ij} - \frac{1}{2}\bar{h}\eta_{ij} \tag{48}$$

With $\bar{h}_{00}$ the only non-negligible term, we get

$$\bar{h} = \bar{h}_{00}\eta^{00}h_{00} = \frac{1}{2}\bar{h}_{00} \tag{49}$$

$$h_{ab} = 0 - \frac{1}{2}\bar{h}\eta_{ab} = \frac{1}{2}\bar{h}_{00}\delta_{ab} \tag{50}$$

where a, b take values of 1 to 3.