Physics 530-11

Penrose diagram of collapse

The Vaiyda metric

$$ds^2 = (1 - \frac{2M(v)}{r})dv^2 - 2dvdr - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \tag{1}$$

is the solution to Einstein's equations of a collapsing null dust (ie matter whose stress energy tensor is of the pressureless form

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{2}$$

where in this case the velocity u^μ is a null vector, rather than a timelike velocity. In our case, we can take u^μ to be the r coordinate axis, and ρ in that case is proportional to $\partial_v M(v)/r^2$.

If we choose $M(v)$ to be a step function $M(v) = 0 \quad v < 0$ and $M(v) = M_0 > 0 \quad v > 0$ then the infalling null matter is a shell like (delta function) sphere of matter. Thus we have

$$ds^2 = dv^2 - 2dvdr - r^2 d\Omega^2 \qquad v < 0 \tag{3}$$

$$ds^2 = (1 - \frac{2M_0}{r})dv^2 - 2dvdr + r^2 d\Omega^2 \qquad v > 0 \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin(\theta)^2 d\phi^2$. Going to null coordinates, we can define the coordinate u to be $v - 2r$ for r¡0. This gives the metric

$$ds^2 = dvdu - (\frac{v-u}{2})^2 d\Omega^2 \tag{5}$$

For $v > 0$ we want the coordinate u to be continuous (which implies that $r(v,u)$ must be continuous as a function of v) This can be done by defining

$$f(u) = (v - 2(r + 2M_0 \ln(\frac{r}{2M_0} - 1)) \tag{6}$$

for some function f. Thus, we must have that

$$(-2(r + 2M_0 \ln(\frac{r}{2M_0} - 1)) = f(-2r) \tag{7}$$

or

$$f(u) = u - 4M_0 \ln(\frac{-u}{4M} - 1) \tag{8}$$

Note that this works only for $u < -4M$. We then have for $v > 0$

$$ds^2 = \frac{2M_0}{r}(\frac{r}{2M_0} - 1)f'(u)dvdu - r^2 d\Omega^2 \tag{9}$$

where r is taken as a function if v, u defined by

$$r + 2M_0 \ln(\frac{r}{2M_0} - 1) = \frac{v - f(u)}{2} \tag{10}$$

The metric for $v > 0$ thus becomes

$$ds^2 = \frac{r}{2M_0} e^{(v-f(u))/4M - r/2M}(1 - \frac{4M_0}{-u - 4M_0})dudv - r(v,u)^2 d\Omega^2 \tag{11}$$

$$= \frac{r(v,u)}{2M_0} e^{\frac{-r(v,u)}{2M_0}} e^{(v-u)/4M_0} dvdu - r(v,u)^2 d\Omega^2 \tag{12}$$

$r(v, u)$ is a continuous function of v, u for all values of v, u, and thus this metric is a continuous non-degenerate, non-singular function of v, u.

The Penrose confomal transformation is obtained taking $v = tan(V)$ and $u = \tan(U)$, and multiplying the resultant metric by $\cos^2(V)\cos^2(U)$ (ie making a confomal transformation). This gives us

$$d\hat{s}^2 = \frac{r(v,u)}{2M_0} e^{\frac{-r(v,u)}{2M_0}} e^{(v-u)/4M_0} dVdU - r(v,u)^2 \cos^2(V)\cos^2(U) d\Omega^2 \tag{13}$$

We have

$$r(v, u) + 2M_0 \ln((r/2M_0) - 1_{=}(v - f(u))/2 = (v - u)/2 - 2M_0 ln(-1 - u/4M_0) \tag{14}$$

which implies that as $u \to -4M_0$,

$$r \approx 2M_0 + e^{v/2}(u + 4M_0) \tag{15}$$

while for large r

$$r\cos(V)\cos(U) \approx sin(V - U) \tag{16}$$

Thus, the metric $d\hat{s}^2$ is a finite metric everywhere, with $-\frac{\pi}{2} < V < \frac{\pi}{2}$ and $-\frac{\pi}{2} < U < 0$

The conformal diagram is given in Figure 1. $\mathcal{J}^+$ and $\mathcal{J}^-$ are future and past null infinity ($r = \infty$). I^+ is the singular (in the conformal space) future timelike infinity while I^- is past timelike infinity. I^0 is spacelike infinity. All of the I are singular points in the conformal metric $d\hat{s}^2$.

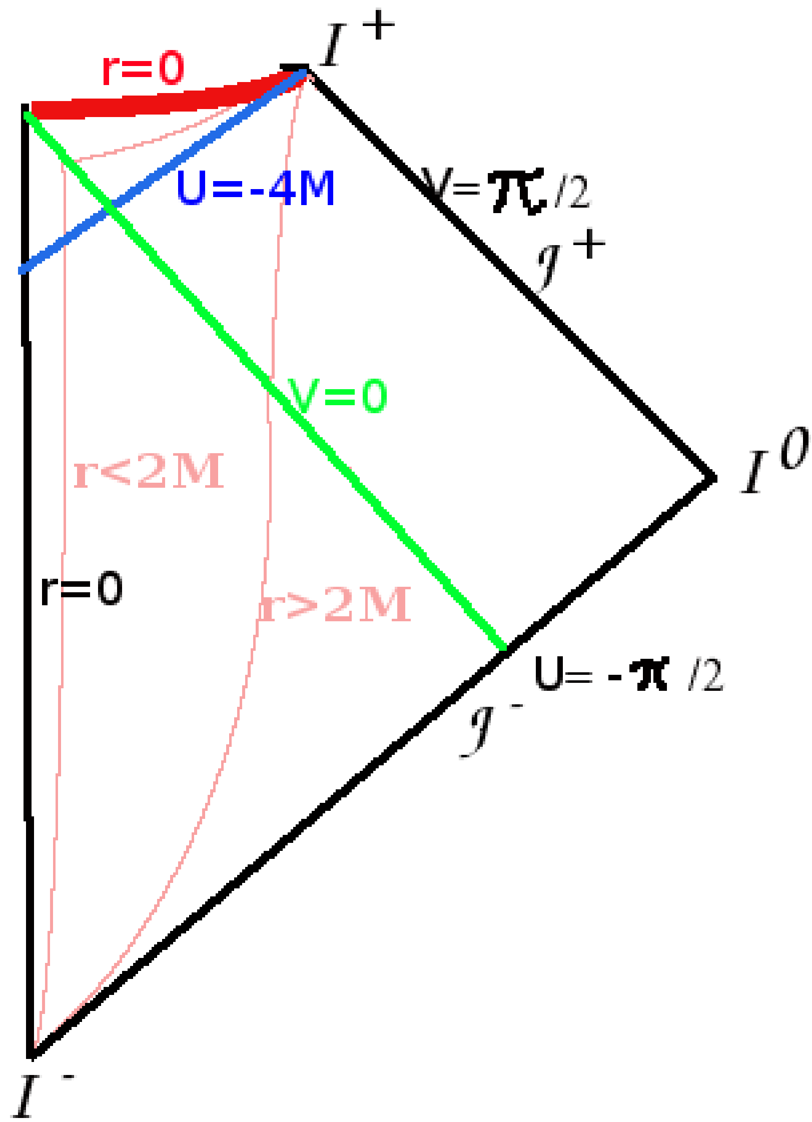

Figure 1: The Penrose conformal diagram for the black hole created by the collapse of a null shell of fluid.