Gravitational waves in general relativity
III. Exact plane waves

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Plane gravitational waves are here defined to be non-flat solutions of Einstein's empty space-time field equations which admit as much symmetry as do plane electromagnetic waves, namely, a 5-parameter group of motions. A general plane-wave metric is written down and the properties of plane wave space-times are studied in detail. In particular, their characterization as 'plane' is justified further by the construction of 'sandwich waves' bounded on both sides by (null) hyperplanes in flat space-time. It is shown that the passing of a sandwich wave produces a relative acceleration in free test particles, and inferred from this that such waves transport energy.

1. Introduction

The theory of gravitational waves was initiated by Einstein (1917, 1919) in two papers on the approximate wave-like solutions of his empty-space field equations. Subsequently Einstein & Rosen (1937) investigated plane and cylindrical gravitational waves, and Rosen (1937) came to the conclusion that there were no exact plane-wave metrics filling all space-time. Rosen's result was that in any non-flat solution of the empty space field equations satisfying the symmetry conditions for plane waves which he had imposed, there had to be a 2-space on which the determinant of the metric tensor vanished. He rejected such solutions as unphysical.

However, Rosen demanded what seems unnecessarily severe, although at the time it was usual, namely, that the whole of space-time be covered by one non-singular co-ordinate system. Such conditions could not be satisfied on the surface of a sphere in ordinary Euclidean 3-space. In effect, Rosen did not distinguish sufficiently between co-ordinate singularities (like the singularity at the origin of polar co-ordinates) and physical singularities, which could, in principle, be discovered experimentally. At the time, the mathematical foundation of general relativity were not well developed, but they have since been put into good order by Lichnerowicz, in a series of papers now collected into a treatise (Lichnerowicz 1955). It is no longer usual to impose the conditions for regularity employed by Rosen. Lichnerowicz has devised less stringent conditions which the co-ordinate system and the metric should be expected to satisfy if the space-time is to be physically reasonable and mathematically respectable (Lichnerowicz 1955, chap. I). In the following, we shall assume Lichnerowicz's conditions, without further

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attempts to justify them. By these criteria, Rosen's plane-wave metrics are physically and mathematically acceptable, as was discovered independently by two of us (Robinson 1956, unpublished; Bondi 1957; see also, Bonnor 1957). In the meantime, a different application of Lichnerowicz's conditions, with some other considerations, led (Pirani 1957, referred to in the following as RT) to a general criterion of gravitational radiation, which the plane-wave metrics indeed satisfy.

Unfortunately, plane gravitational waves do not exhibit their planeness in so clear a way as plane electromagnetic waves do, and the published plane-wave solutions have received some criticism. We have therefore thought it necessary to discuss plane-wave metrics in detail in this paper, and to investigate their physical properties more thoroughly than hitherto. Our interest in plane waves derives not, of course, from the expectation that such waves might exist in nature, but from the presumption that at great distances from a finite source of gravitational waves, these waves must appear to be approximately plane. The plane-wave metrics to be discussed below satisfy stringent symmetry conditions. More general solutions of Einstein's equations, which have less symmetry, but which are plane in the sense that the surfaces of constant phase are planes, will be discussed elsewhere.

In §2, plane gravitational waves are defined rigorously and the plane-wave metrics are written down. In §3 the planeness of the waves is discussed. In §4, other physical properties of the waves are discussed in terms of the Riemann tensor. Approximate plane-wave solutions and combined gravitational–electromagnetic plane waves will be discussed in further papers.

2. Definition of plane waves

We shall assume that plane gravitational waves are represented by non-flat solutions of Einstein's empty space-time field equations

$$G_{\mu\nu} = 0, \tag{2·1}$$

where $G_{\mu\nu}$ is the Einstein tensor. (Throughout this paper the conventions are that Greek indices range and sum over 0, 1, 2, 3, that the metric tensor has signature -2, and that $c = 1$.)

This ensures that the waves are purely gravitational, not electromagnetic or hydrodynamic. To ensure that they are in fact waves, and that they are plane, we shall require that they possess a degree of symmetry analogous to that possessed by plane electromagnetic waves in flat space-time. There are some advantages in this indirect approach, as opposed to an attempt to extract formal definitions immediately from our physical intuitions, because so little is known about rigorous solutions of Einstein's equations. Of course there are metrics which exhibit a certain 'plane' symmetry without having anything to do with waves (for some examples, see Taub 1951), but we shall see shortly that the symmetry of plane electromagnetic wave fields is so high that the demand for corresponding symmetry of gravitational fields will automatically ensure their wave-like character. The wave-like character will appear in the intrinsically *null* properties of these metrics. These will not be 'co-ordinate waves' whose metrics can be transformed into static form by a co-ordinate transformation (cf. McVittie 1955). The nullness will become apparent from the physical properties of the metrics, and we shall make it manifest in an invariant way by exhibiting the geometrical properties of the Riemann tensor, which we shall show to be of the type previously associated with radiation (cf. RT).

To study the invariance properties of plane electromagnetic waves, let us fix our attention on plane waves propagated in the direction of the positive x-axis. Their obvious symmetry property is that the field is the same at every point of a wave-front. In more formal language, we may say that there is a 3-parameter group of motions of the Minkowski space-time into itself leaving the electromagnetic field unaltered. The motions are translations in the y- and z-directions and along the null 3-surfaces $t-x=$ constant, which are the invariant subspaces of the group.

Besides these obvious symmetries, there are additional symmetries of plane electromagnetic waves which are less obvious because they are intrinsically 4-dimensional. These symmetries are described by a further 2-parameter group of motions taking the null 3-surfaces $t-x=$ constant into themselves. These motions might well be called 'null rotations', since in contrast to other, more familiar transformations of the homogeneous Lorentz group, which leave two 2-spaces (one time-like, one space-like) unaltered, these transformations leave one null 3-space unaltered. (Null rotations have been discussed by Shibata: 1955 and earlier papers referred to there.)

For an electromagnetic plane wave in flat space-time we may write the field in the form (cf. Synge, 1956, pp. 350–353)

$$F_{\mu\nu} = B(u)\left[(k_\mu l_\nu - k_\nu l_\mu)\cos\theta(u) + (k_\mu m_\nu - k_\nu m_\mu)\sin\theta(u)\right], \tag{2·2}$$

where k_μ is the (constant) propagation vector ($k_\mu k^\mu = 0$), l_μ and m_μ are constant space-like vectors orthogonal to $k_\mu(k_\mu l^\mu = k_\mu m^\mu = 0)$ and, for convenience, to each other ($l_\mu m^\mu = 0$), which may without loss of generality be taken to be unit vectors ($l_\mu l^\mu = m_\mu m^\mu = -1$). The amplitude and polarization of the wave are described, respectively, by the arbitrary functions $B(u)$ and $\theta(u)$ of the argument $u = k_\mu x^\mu$. For waves propagated in the positive x-direction we may in a Minkowski co-ordinate system (with

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu)$$

take $k^\mu = \delta^\mu_0 + \delta^\mu_1$ and l^μ and m^μ along the y and z axes, respectively: $l^\mu = \delta^\mu_2$, $m^\mu = \delta^\mu_3$ (here and elsewhere δ^μ_ν is the Kronecker delta). The field tensor $F_{\mu\nu}$ given by (2·2) is a function of $k_\mu x^\mu = u = t-x$ only. The tensor $F_{\mu\nu}$ is invariant under a 5-parameter subgroup of the inhomogeneous Lorentz group, comprising the transformations

$$\left.\begin{aligned} x'^0 &= x^0 + a^0 + b^k(x^k + \tfrac{1}{2}a^k + \tfrac{1}{2}b^k u),\\ x'^1 &= x^1 + a^1 + b^k(x^k + \tfrac{1}{2}a^k + \tfrac{1}{2}b^k u),\\ x'^k &= x^k + a^k + b^k u, \end{aligned}\right\} \tag{2·3}$$

where Latin indices k range and sum over 2, 3 only, and a^μ, with $a^0 = a^1$, and b^k are together the 5 independent parameters. The product of two transformations is another of the same type. The transformations T_a defined by setting $b^k = 0$ form the 3-parameter group of translations mentioned above. The null rotations are the transformations T_b defined by setting $a^\mu = 0$. The infinitesimal generators $\xi_A{}^\mu$, $A = 1, 2, \ldots, 5$, of these transformations are

$$T_a\begin{cases} a^0\colon \xi_1{}^\mu = (1, 1, 0, 0), \\ a^2\colon \xi_2{}^\mu = (0, 0, 1, 0), \\ a^3\colon \xi_3{}^\mu = (0, 0, 0, 1), \end{cases} \qquad T_b\begin{cases} b^2\colon \xi_4{}^\mu = (y, y, u, 0) = x^2\xi_1{}^\mu + u\xi_2{}^\mu, \\ b^3\colon \xi_5{}^\mu = (z, z, 0, u) = x^3\xi_1{}^\mu + u\xi_3{}^\mu. \end{cases} \tag{2·4}$$

The structure constants of the transformation group are all zero except for

$$C^1_{24} = 1, \quad C^1_{35} = 1. \tag{2·5}$$

It will be useful later on to know the effect of a null rotation on the world-line of the space origin $x^1, x^2, x^3 = 0$. The transformations T_b take this line into the line

$$x^0 = (1 + \tfrac{1}{2}b^k b^k)\,\tau, \quad x^1 = \tfrac{1}{2}b^k b^k \tau, \quad x^k = b^k \tau,$$

where τ is proper time along the line. This line represents a motion with uniform velocity whose Newtonian components are

$$\mathbf{v} = (1 + \tfrac{1}{2}b^k b^k)^{-1}\,(\tfrac{1}{2}b^k b^k, b^2, b^3). \tag{2·6}$$

With this information about the symmetry of plane electromagnetic waves, we turn now to the definition of plane gravitational waves. We have assumed already that

(A) *a plane-wave metric is a non-flat solution of the empty space-time equations.*

We assume now in addition that

(B) *a plane-wave metric admits a* 5*-parameter group of motions.*

It is not necessary to assume anything about the structure of this group, nor to specify that its invariant subspaces be null 3-spaces; this will appear automatically.

Fortunately, we do not have to investigate empty space-time metrics admitting groups of motions. This laborious task has been undertaken by Petrov (1957), who has enumerated all such metrics. According to his results (quoted in Petrov 1955), no empty space-time except Minkowski space-time admits a group with more than 6 parameters. There are several empty space-times admitting 6-parameter groups; from the present point of view, they may be regarded as special cases of the class of empty space-times admitting a 5-parameter group. It is always possible to reduce the metric of this class in a finite region (though not necessarily throughout the space-time) to the form, sufficiently general for our purposes,

$$\mathrm{d}s^2 = \exp(2\phi)\,(\mathrm{d}\tau^2 - \mathrm{d}\xi^2) - u^2[\cosh 2\beta(\mathrm{d}\eta^2 + \mathrm{d}\zeta^2) + \sinh 2\beta \cos 2\theta(\mathrm{d}\eta^2 - \mathrm{d}\zeta^2) - 2\sinh 2\beta \sin 2\theta\, \mathrm{d}\eta\, \mathrm{d}\zeta] \tag{2·7}$$

given previously (Bondi 1957). This form is equivalent locally to that given by Petrov, and more convenient for calculations. Here ϕ, β, and θ are functions of $u = \tau - \xi$, which must in empty space-time satisfy the condition

$$2\phi' = u(\beta'^2 + \theta'^2 \sinh^2 2\beta). \tag{2·8}$$

The function ϕ is determined by this equation, β and θ being given arbitrarily as functions of u. The space-time is not flat unless

$$\left.\begin{aligned}\sigma &\equiv u^{-2}\exp(-2\phi)\,[\beta''+2u^{-1}\beta'-u\beta'^3-2\theta'^2\sinh 2\beta\cosh 2\beta \\ &\qquad -u\beta'^2\theta'^2\sinh^2 2\beta] = 0,\\ \omega &\equiv -u^{-2}\exp(-2\phi)\,[\theta''\sinh 2\beta+\theta'\{4\beta'\cosh 2\beta+2u^{-1}\sinh 2\beta-u\sinh 2\beta \\ &\qquad \times(\beta'^2+\theta'^2\sinh^2 2\beta)\}] = 0.\end{aligned}\right\} \tag{2·9}$$

Roughly speaking, β defines the amplitude of the wave, and θ its direction of polarization (see §4). A simple special case which exhibits most of the interesting features of the waves is the case of fixed plane of polarization, $\theta = 0$. In this case (2·7), (2·8) and (2·9) reduce to

$$\mathrm{d}s^2 = \exp(2\phi)\,(\mathrm{d}\tau^2-\mathrm{d}\xi^2)-u^2\{\exp(2\beta)\,\mathrm{d}\eta^2+\exp(-2\beta)\,\mathrm{d}\zeta^2\}, \tag{2·7$'$}$$

$$2\phi' = u\beta'^2, \tag{2·8$'$}$$

and the single flatness condition

$$\beta''+2u^{-1}\beta'-u\beta'^3 = 0. \tag{2·9$'$}$$

Because β is a completely arbitrary function of u, we may fix the amplitude of the waves at will. It is not necessary to consider a space-time which is everywhere filled with radiation. It is simpler and more illuminating to consider waves of finite duration, 'sandwich' waves, with amplitude non-zero only for a finite range of u (not including $u = 0$) in the 'filling' of the sandwich. Elsewhere, the space-time is flat. Such a situation is permissible, because Lichnerowicz's conditions do not require that the metric tensor components be analytic functions of the co-ordinates, or that one co-ordinate system cover all space-time.

In fact, space-time cannot in general be covered entirely by the co-ordinate system S_1: (τ, ξ, η, ζ), since the metric (2·7) or (2·7′) becomes singular when $u = 0$. However, for a sandwich wave, matters may be arranged as shown in figure 1. In the filling C, and for a finite range A, B, D, E, of u on either side of it, the co-ordinate system is S_1 with metric (2·7). The filling C divides space-time into two flat regions, and in each flat region we may introduce an ordinary Minkowskian co-ordinate system S_2: (t, x, y, z) which overlaps the system S_1 in the flat outer layers A and E of the sandwich. Then it is easy to arrange matters so that β, θ and ϕ are sufficiently continuous, and that the transformation from S_1 to S_2 in A and E is sufficiently differentiable, and then so long as the sandwich is bounded away from $u = 0$, all Lichnerowicz's conditions will be satisfied. The transformations from S_1 to S_2 can become very complicated, so we shall discuss them only for the case $\theta = 0$. Then the flatness condition (2·9′) and the equation (2·8′) determining ϕ may be satisfied by

$$\left.\begin{aligned}&\text{(i)}\quad \beta=\beta_0,\quad \phi=\phi_0;\\ &\text{(ii)}\quad \beta=\beta_0+\log u,\quad \phi=\phi_0+\tfrac{1}{2}\log u;\\ \text{or}\quad &\text{(iii)}\quad \beta=\beta_0+\log|\{(1\pm u^2/u_0^2)^{\frac{1}{2}}-1\}\{u/u_0\}^{-1}|,\\ &\qquad \phi=\phi_0+\tfrac{1}{2}\log\{u/(1\pm u^2/u_0^2)^{\frac{1}{2}}\}.\end{aligned}\right\} \tag{2·10}$$

Here β_0, ϕ_0 and u_0 are arbitrary constants. The metric (2·7′) may in each case be transformed to the Minkowskian form. The transformations are

$$
\left.\begin{array}{ll}
\text{(i)} & \begin{cases} \tau-\xi = t-x = u, \\ \tau+\xi = \exp(-2\phi_0)\{t+x-u^{-1}(y^2+z^2)\}, \\ \quad \eta = \exp(-\beta_0)\,u^{-1}y, \quad \zeta = \exp(\beta_0)\,u^{-1}z; \end{cases} \\
\text{(ii)} & \begin{cases} \tau-\xi = \{2(t-x)\}^{\frac{1}{2}} = u, \\ \tau+\xi = \exp(-2\phi_0)\{t+x-(t-x)^{-1}y^2\}, \\ \quad \eta = \exp(-\beta_0)\,u^{-2}y, \\ \quad \zeta = \exp(\beta_0)\,z; \end{cases} \\
\text{and (iii)} & \begin{cases} \tau-\xi = |(t-x)^2-u_0^2|^{\frac{1}{2}} = u, \\ \tau+\xi = \exp(-2\phi_0)\,u_0^{-1}\{t+x-(t-x-u_0)^{-1}y^2-(t-x+u_0)^{-1}z^2\}, \\ \quad \eta = \exp(-\beta_0)\,(t-x-u_0)^{-1}y, \quad \zeta = \exp(\beta_0)\,(t-x+u_0)^{-1}z. \end{cases}
\end{array}\right\} \quad (2·11)
$$

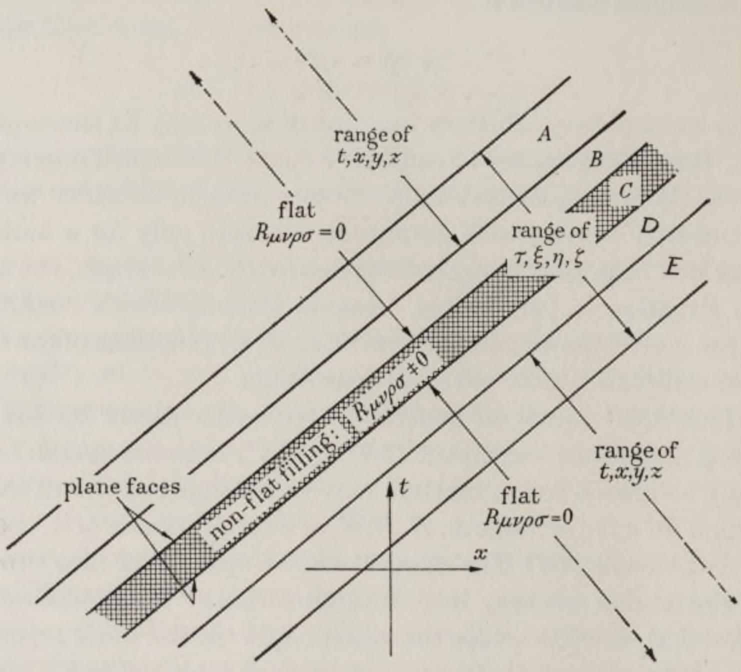

FIGURE 1. Arrangement of co-ordinate systems around sandwich wave.

In the co-ordinate system S_1: (τ, ξ, η, ζ), with metric (2·7), the generators of the group of motions are given by

$$
\left.\begin{aligned}
\xi_1{}^\mu &= \delta^\mu{}_0+\delta^\mu{}_1, \\
\xi_2{}^\mu &= \delta^\mu{}_2, \\
\xi_3{}^\mu &= \delta^\mu{}_3, \\
\xi_4{}^\mu &= \eta\xi_1{}^\mu+P_-(u)\,\xi_2{}^\mu+N(u)\,\xi_3{}^\mu, \\
\xi_5{}^\mu &= \zeta\xi_1{}^\mu+P_+(u)\,\xi_3{}^\mu+N(u)\,\xi_2{}^\mu,
\end{aligned}\right\} \quad (2·12)
$$

where
$$\left.\begin{aligned} P_{\pm}(u) &= \int^{u} u^{-2}\exp(2\phi)(\cosh 2\beta \pm \sinh 2\beta\cos 2\theta)\,\mathrm{d}u, \\ N(u) &= \int^{u} u^{-2}\exp(2\phi)\sinh 2\beta\sin 2\theta\,\mathrm{d}u. \end{aligned}\right\} \tag{2·13}$$

All the corresponding operators commute except for

$$[X_2, X_4] = X_1, \quad [X_3, X_5] = X_1. \tag{2·14}$$

Accordingly, the structure constants are the same as for the transformation group (2·5) of plane electromagnetic waves. This is partly accidental. The two sets of generators can be compared sensibly only in flat space-time, and the comparison will depend on which transformation from S_1 to S_2 leads to Minkowskian co-ordinates. For example, if we apply the transformation (2·11) (i) to the generators (2·12) with $\theta = 0$, $\beta_0 = \phi_0 = 0$, so that $P_{\pm}(u) = -u^{-1}$, $N(u) = 0$, then in S_2 we obtain

$$\left.\begin{aligned} &\xi_1{}^{\mu} = \delta^{\mu}{}_0 + \delta^{\mu}{}_1, \quad \xi_2{}^{\mu} = y(\delta^{\mu}{}_0 + \delta^{\mu}{}_1) + u\delta^{\mu}{}_2, \\ &\xi_3{}^{\mu} = z(\delta^{\mu}{}_0 + \delta^{\mu}{}_1) + u\delta^{\mu}{}_3, \quad \xi_4{}^{\mu} = -\delta^{\mu}{}_2, \quad \xi_5{}^{\mu} = -\delta^{\mu}{}_3. \end{aligned}\right\} \tag{2·15}$$

The correspondence from (2·5) to (2·15) is not $\xi_A{}^{\mu} \to \xi_A{}^{\mu}$, $A = 1, 2, \ldots, 5$, but rather $\xi_1{}^{\mu} \to \xi_1{}^{\mu}$, $\xi_2{}^{\mu} \to -\xi_4{}^{\mu}$, $\xi_3{}^{\mu} \to -\xi_5{}^{\mu}$, $\xi_4{}^{\mu} \to \xi_2{}^{\mu}$, $\xi_5{}^{\mu} \to \xi_3{}^{\mu}$. The changes in sign account for the fact that the commutators are the same, although two pairs of generators have been exchanged. If we were to apply instead the transformation (2·11) (ii) or (2·11) (iii), a different correspondence would result. The physical significance of these transformations will be discussed further in the next section.

3. Planeness of plane waves

The main difficulty in the understanding of gravitational plane waves is that they are not as plane as might have been expected. The difficulty becomes very clear if we try to write the plane-wave metric (2·7′) in a form which exhibits its departure from Minkowskian character. For example, if we make the transformation (cf. Bondi 1957)

$$\left.\begin{aligned} \tau - \xi &= t - x = u, \\ \tau + \xi &= \exp(-2\phi)\{t + x - u^{-1}(y^2 + z^2)\}, \\ \eta &= \exp(-\beta)\,u^{-1}y, \quad \zeta = \exp(\beta)\,u^{-1}z, \end{aligned}\right\} \tag{3·1}$$

which reduces to (2·11) (i) when β and ϕ are constants, the metric (2·7′) becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - \{(t^2 - x^2)\beta'^2 + 2u^{-1}(y^2 - z^2)\beta'\}\,\mathrm{d}u^2 + 2\beta'(y\,\mathrm{d}y - z\,\mathrm{d}z)\,\mathrm{d}u. \tag{3·2}$$

Departures from flatness are represented entirely by the variability of the arbitrary function β. However, the metric tensor no longer depends on the single variable $u = t - x$; y and z appear in it explicitly. Even if β and its derivatives are very small, wherever they do not vanish, the departures from Minkowskian values will become large for large y and z. Large departures can be prevented by making different transformations in each of an infinite number of overlapping regions, but so long as it is required that departures from flatness be represented against a Minkowskian background, the y and z dependence cannot be avoided. This has led a number of critics to insist that the metrics (2·7) do not deserve the name 'plane-wave metrics'.

However, the concept of planeness, whatever intuitive ideas one may have of it, must be formulated in a precise manner before one can decide whether it applies to a particular case. We have chosen to define the planeness of gravitational waves by demanding that they have as much symmetry as do plane electromagnetic waves. In each case the symmetry is described by a motion group, and the two motion groups have similar structure, as we have seen, although the exact correspondence depends on the individual case. We know from the results of Petrov that no other empty space-time metrics possess the same degree of symmetry. And as we have shown in the preceding section, we can construct sandwich waves bounded by plane faces, faces plane, moreover, in Minkowski space-time, where there can be no doubt about their planeness.

Thus it seems unlikely that there are any metrics planer than these. Why then does the dependence on the space-variables insist on showing itself if we try to write the metrics in quasi-Minkowskian form? It is essentially this fact which contradicts our intuitive ideas of planeness. We shall try to explain it by investigating some of the physical properties of plane-wave metrics.

Let us consider to *whom* a plane-wave metric may appear plane. We shall approach this question by analogy, through a discussion of the corresponding question in relativistic cosmology.

In cosmology, one usually assumes that the universe is homogeneous and isotropic. Space-time, which is non-empty, is required to be invariant under a 6-parameter group consisting of translations and rotations of the invariant hypersurfaces cosmic time = constant. The metric may be written in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\,\{1+\tfrac{1}{4}k(x^2+y^2+z^2)\}^{-2}\,(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2). \tag{3·3}$$

Further assumptions are required to determine the function $R(t)$ and the constant k.

To whom does such a universe appear homogeneous and isotropic? Only a 'substratum' observer with fixed x, y and z, moving along a time-line, will find it so. An observer moving relative to the substratum will observe an angular variation in the luminosity/red shift relation and in other observable quantities. Of course, he will be able to *deduce* from the totality of his observations that the universe is in fact homogeneous and isotropic and only appears otherwise because of his own peculiar motion. But only those observers who move with the substratum will immediately observe the symmetry of the universe. These observers may be distinguished from all others by the special relation between their 4-velocities and the generators of the group of motions, namely, that the 4-velocities are orthogonal to all the generators. Moreover, these observers are *equivalent* to one another in the sense that the world-line of any one of them may be taken into the world-line of any other by a suitably chosen motion of the group. It follows that all the observations they can make and all the conclusions they will draw must agree, item by item.

Thus may we draw from cosmology two useful concepts: the concept of an observer moving in a special relation to the symmetry properties of space-time, and the concept of equivalent observers whose world-lines can be transformed into one another by the motions of the symmetry group.

Let us apply these ideas to the plane-wave space-times. Of course the concept of being in a special relation is not quite well defined; we shall have to appeal to physical intuition to decide which relationships are 'special'. Fortunately, by considering sandwich waves we can make the appeal in flat space-time where intuition is less likely to be fallible.

Consider, for example, a sandwich wave with fixed plane of polarization, and flatness condition satisfied in the form (2·10) (i) on either side of the filling, say

$$\left.\begin{aligned} &0 < u \leqslant u_1\colon \beta = 0, \quad \phi = 0; \\ &u_1 < u_2 \leqslant u\colon \beta = 0, \quad \phi = \tfrac{1}{2}a; \quad a \text{ constant,} \end{aligned}\right\} \tag{3·4}$$

with β varying arbitrarily in the range $u_1 \leqslant u \leqslant u_2$, and ϕ determined by (2·8′), namely $\phi' = \frac{1}{2}u\beta'^2$, from which it follows that $a > 0$.

We shall determine the effect of this wave on a family of observers who before its arrival are relatively at rest in a Minkowskian inertial frame. Let us fix attention on observers who are at rest in the Minkowskian co-ordinate system S_2. Their 4-velocities are orthogonal to $\xi_4{}^\mu$ and $\xi_5{}^\mu$ and have constant scalar product with $\xi_1{}^\mu$ but stand in no simple relation to $\xi_2{}^\mu$ or $\xi_3{}^\mu$. If we assume that these observers move freely during the passing of the wave, so that their world-lines are geodesics, then these relations will persist, because the equations of geodesics always admit first integrals $\xi_{A\mu}(\mathrm{d}x^\mu/\mathrm{d}s) = \text{constant}$.

These observers are equivalent to one another, and all have the same experience of the passing of the wave. The translations generated by $\xi_1{}^\mu$, $\xi_4{}^\mu$ and $\xi_5{}^\mu$ take their world-lines into one another. Yet although they are relatively at rest before the wave arrives, they will be relatively in motion after it has passed. For consider first the geodesic observer O who is initially at rest at the origin of the Minkowskian co-ordinate system S_2. In the co-ordinate system S_1, connected to S_2 by (2·11) (i), his world-line has the equations

$$\tau + \xi = s, \quad \Phi(\tau - \xi) = s, \tag{3·5}$$

where $\Phi(u) = \int^u \exp(2\phi)\,\mathrm{d}u$, the constant of integration being chosen so that $\Phi(u) = u$ for $u \leqslant u_1$. After the wave has passed, we have $\Phi(u) = \exp(a)\,u + h$, where h is a constant determined by the behaviour of $\phi(u)$ in the filling. Transforming back to the Minkowskian system S_2 by (2·11) (i) we find for O's world-line the equations $t = s\cosh a - \frac{1}{2}h\exp(-a)$, $x = s\sinh a + \frac{1}{2}h\exp(-a)$. The displacement and velocity of O have no invariant significance, since the transformation (2·11) (i) is determined only up to a Lorentz transformation, and O may be brought back to rest at the space origin by the transformation

$$\begin{aligned} t' &= \{t + \tfrac{1}{2}h\exp(-a)\}\cosh a - \{x - \tfrac{1}{2}h\exp(-a)\}\sinh a, \\ x' &= -\{t + \tfrac{1}{2}h\exp(-a)\}\cosh a + \{x - \tfrac{1}{2}h\exp(-a)\}\cosh a, \\ y' &= y,\ z' = z, \end{aligned}$$

to a new Minkowskian system S_2'.

To find the observers O^* equivalent to O we must apply to O's world-line the integrated motions of the group generated by (2·12). In the co-ordinate system S_1 these motions are

$$\begin{aligned}
\xi_1{}^\mu &: \exp(kX_1)\,x^\mu = x^\mu + k\xi_1{}^\mu,\\
\xi_2{}^\mu &: \exp(kX_2)\,x^\mu = x^\mu + k\xi_2{}^\mu,\\
\xi_3{}^\mu &: \exp(kX_3)\,x^\mu = x^\mu + k\xi_3{}^\mu,\\
\xi_4{}^\mu &: \exp(kX_4)\,x^\mu = x^\mu + k(x^2\xi_1{}^\mu + P_-(u)\,\xi_2{}^\mu) + \tfrac{1}{2}k^2P_-(u)\,\xi_1{}^\mu,\\
\xi_5{}^\mu &: \exp(kX_5)\,x^\mu = x^\mu + k(x^3\xi_1{}^\mu + P_+(u)\,\xi_3{}^\mu) + \tfrac{1}{2}k^2P_+(u)\,\xi_1{}^\mu,
\end{aligned}$$

where (in the case of fixed plane of polarization now being considered, $\theta = 0$ and)

$$P_\pm(u) = \int^u u^{-2}\exp(2\phi \pm 2\beta)\,\mathrm{d}u.$$

For the sandwich wave, $P_\pm(u) = p_\pm - \exp(a)\,u^{-1}$ after the passage of the wave; here $p_\pm$ are constants determined by the behaviour of β and ϕ in the filling.

For example, let us apply to O's world-line the motion generated by $\xi_4{}^\mu$. Before the arrival of the wave, this is simply a translation in the negative y-direction. In the Minkowskian system S_2 in which O is at rest at the origin before the arrival of the wave, O^* is at rest at $(0, -k, 0)$. But in the Minkowskian system S_2' in which O is at rest at the origin after the wave has passed, O^*'s world-line is given by

$$\begin{aligned}
t &= s + \tfrac{1}{2}\exp(-2a)\,k^2p_-{}^2\{s - h - \exp(2a)\},\\
x &= \tfrac{1}{2}\exp(-2a)\,k^2p_-{}^2\{s - h - \exp(2a)\},\\
y &= k\{p_-\exp(-a)\,(s-h) - \exp(a)\},\\
z &= 0,
\end{aligned}$$

so that he now has a Newtonian velocity

$$\mathbf{v} = \{1 + \tfrac{1}{2}\exp(-2a)\,k^2p_-{}^2\}^{-1}\,[\tfrac{1}{2}\exp(-2a)\,k^2p_-{}^2, \exp(-a)\,kp_-, 0]$$

relative to O. This is of the same form as (2·6) (with $b^3 = 0$).

Thus the effect of the wave is to develop a relative acceleration between neighbouring freely-moving observers, and thus, cumulatively, a relative velocity. Since the observers are equivalent, it is easy to see from symmetry (or from the above formulae) that the relative velocity must increase with separation. These large relative velocities developing between equivalent observers correspond physically to the y and z terms in a quasi-Minkowskian metric. Of course, the preferred role given to O has no invariant significance, and one could equally well by a Lorentz transformation reach a situation where O^* was at rest before and after the passage of the wave, and O acquired a velocity. Moreover, observers who are relatively at rest after the wave has passed must have been relatively in motion before it arrived. An observer O^* displaced relative to O in the z-direction instead of the y-direction would also acquire a relative velocity, but its magnitude and direction would be different, because p_- would be replaced by p_+, and the roles of y and z interchanged. The difference between p_- and p_+ arises from the polarization of the wave.

As previously pointed out by one of us (Bondi 1957), this relative acceleration and consequent relative velocity prove that gravitational waves transport energy, since it is in principle possible, utilizing this effect, to construct a device which will extract energy from a wave. The simplest such device consists of a stiff rod (the rod need not be rigid in the technical sense, and the difficulties surrounding the consideration of rigid bodies in relativity theory are not relevant here) and a bead which slides on the rod with some friction. If the rod lies in a suitable direction transverse to the direction of wave propagation, and if the bead is at rest relative to the rod at a position well displaced from the rod's centre of mass, the passing of the wave will result in some relative motion of the rod and the bead, for in the first approximation the bead and the mass centre of the rod will each move on a geodesic. This relative motion will generate heat, and thus locally available energy may be extracted from the wave.

In these considerations, the effect of the device on the wave has been neglected. This is a test device—a device constructed out of test particles. Consequently, such considerations cannot be used to calculate the total amount of available energy in the wave.

The accelerating properties of the wave can be discussed concisely in terms of the Riemann tensor of the sandwich region, which will be examined in the next section.

4. Riemann tensor structure of plane waves

In a previous paper (RT), one of us attempted an invariant formulation of the concept of gravitational radiation in terms of the algebraic structure of the Riemann tensor. Although the plane-wave metrics discussed here satisfy the definition of radiation given previously, we have for other reasons reached the conclusion that the definition is too restrictive for most purposes and applies only to pure radiation. Therefore, we begin this section with a revision of the definition, before considering plane-wave metrics specifically.

Definition of radiation

It is known (Ruse 1944, 1946, 1948; Géhéniau & Debever 1956*a*; Géhéniau 1956*b*; Debever 1956*c*, *d*; Petrov 1954, 1955, 1957, and much further literature cited by these authors) that the Riemann tensor, referred to an orthogonal tetrad of unit vectors, may be put into specially simple form by an appropriate choice of this reference tetrad. In empty space-time there are three distinct types of possible Riemann tensor, Petrov's canonical types (Petrov 1954). In RT, radiation was defined to be present whenever the Riemann tensor was of type II or of type III. As we shall see, the Riemann tensors of plane-wave metrics are of type II. However, after further consideration, and discussion with other workers (C. W. Misner 1957, private communication; Trautman 1958), we consider that the definition is too severe, and describes only pure radiation, just as the self-conjugate field, which is the analogue of type II in the electromagnetic field case, corresponds to pure radiation and not to radiation from a system of charges at a finite distance. We need not reject the definition entirely, but merely weaken it, requiring that the Riemann tensor be asymptotically, though not exactly, of type II. This is to say that, far from a material system, we expect that the dominant terms in the Riemann tensor will have the characteristic form of type II terms, although other terms will be present and the exact form may actually be of type I. This still corresponds to the electromagnetic field case, where the dominant terms in the field tensor for a radiating system will have the same form as a self-conjugate field, but additional terms will be present. But just as plane electromagnetic waves, without real sources, constitute a pure radiation field, having exactly the self-conjugate form, so plane gravitational waves, also without real sources, constitute a pure gravitational radiation field, having a Riemann tensor of type II. For the nature of the asymptotic approach, we refer to the elegant work of Trautman (1958).

Canonical form of Riemann tensor

We compute the physical components of the Riemann tensor for the metric (2·7) with variable plane of polarization. In the co-ordinate system S_1: (τ, ξ, η, ζ), a convenient reference tetrad $\lambda_{(\alpha)}{}^{\mu}$ (α labels the different vectors, μ the components of each) is

$$\left.\begin{aligned} \lambda_{(0)}{}^{\mu} &= \{\exp(-\phi), 0, 0, 0\}, \\ \lambda_{(1)}{}^{\mu} &= \{0, \exp(-\phi), 0, 0\}, \\ \lambda_{(2)}{}^{\mu} &= \{0, 0, u^{-1}\exp(-\beta)\cos\theta, -u^{-1}\exp(-\beta)\sin\theta\}, \\ \lambda_{(3)}{}^{\mu} &= \{0, 0, u^{-1}\exp(\beta)\sin\theta, u^{-1}\exp(\beta)\cos\theta\}. \end{aligned}\right\} \tag{4·1}$$

The Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = -\tfrac{1}{2}\delta^{\alpha\beta}_{\mu\nu}\delta^{\gamma\delta}_{\rho\sigma}\{g_{\alpha\gamma,\beta\delta} + g^{\pi\tau}[\alpha\gamma,\pi][\beta\delta,\tau]\}, \tag{4·2}$$

where $\delta^{\alpha\beta}_{\mu\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}$, and $[\alpha\gamma, \pi]$ are Christoffel symbols of the first kind. The physical components are

$$R_{(\alpha\beta\gamma\delta)} = \lambda_{(\alpha)}{}^{\mu}\lambda_{(\beta)}{}^{\nu}\lambda_{(\gamma)}{}^{\rho}\lambda_{(\delta)}{}^{\sigma}R_{\mu\nu\rho\sigma}. \tag{4·3}$$

We write them in the 6-dimensional formalism, relabelling the index pairs $\alpha\beta$, $\gamma\delta$ according to the scheme

$$\left.\begin{matrix} \alpha\beta: & 23 & 31 & 12 & 10 & 20 & 30 \\ A: & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\}. \tag{4·4}$$

The physical components may then be written in the symmetric array

$$\{R_{AB}\} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\sigma & -\omega & \cdot & -\omega & \sigma \\ \cdot & -\omega & \sigma & \cdot & \sigma & \omega \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\omega & \sigma & \cdot & \sigma & \omega \\ \cdot & \sigma & \omega & \cdot & \omega & -\sigma \end{bmatrix}, \tag{4·5}$$

where σ and ω are defined by (2·9) above. This is of type II, with both scalar invariants zero. Neither σ nor ω is invariant; ω may be eliminated by a rotation of the reference tetrad through an angle $\tan^{-1}(\omega/\sigma)$ in the 23-plane, and the new σ may be given any arbitrary non-zero value by a Lorentz-rotation of the tetrad in the 01-plane. In the case of fixed plane of polarization, $\theta = 0$, and so $\omega = 0$. In this case, (4·5) is already in canonical form; in the general case, it goes into canonical form by the rotation which reduces ω to zero.

The physical significance of the Riemann tensor is (cf. Pirani 1956) that it gives the relative acceleration of free particles, through the equation of geodesic deviation

$$\frac{\delta^2\eta^\mu}{\delta s^2} + R^\mu{}_{\nu\rho\sigma} v^\nu \eta^\rho v^\sigma = 0. \tag{4·6}$$

Here v^ν is the unit tangent vector to a geodesic, and η^μ the orthogonal infinitesimal displacement vector from this geodesic to a neighbouring one (cf. Synge & Schild 1949). The interpretation is reached most easily by referring this equation to a tetrad of which the tangent vector v^ν is the time-like member, and the other members are space-like vectors defined by parallel propagation along the chosen geodesic. Referred to this tetrad, equations (4·6) are the analogues of the Newtonian equations for relative acceleration of neighbouring particles in a gravitational field with scalar potential V:

$$\frac{\mathrm{d}^2X^a}{\mathrm{d}t^2} + K^a{}_b(t)\,X^b = 0, \quad K^a{}_b = \frac{\partial^2 V}{\partial x^a\,\partial x^b}. \tag{4·7}$$

In the case of fixed plane of polarization, the off-diagonal terms of $K^a{}_b$ will be zero; $K^1{}_1$ will be zero and $K^2{}_2$ and $K^3{}_3$ equal and opposite. Referred to this tetrad, the relative accelerations in the y- and z-directions will therefore be independent and in opposite senses, and there will be no relative acceleration in the x-direction, which exhibits the transverse character of the waves. In the general case, the cross-term $K^2{}_3$ is also present.

The vanishing of ω with θ demonstrates clearly the dependence on θ of the directional properties of the wave; the tetrad directions for which ω vanishes define the 'directions of polarization' of the wave in just the same way that the direction of **E** and **H** define the direction of polarization of an electromagnetic wave. The effective integrated amplitude of the wave may be defined in terms of the relative velocity acquired by free particles at unit separation during the passage of the wave; so defined it depends on the functions $P_\pm(u)$ and $N(u)$ of § 2, but no neat expressions for integrated amplitude seem to come out.

Cylindrical and spherical waves

Marder (1958*a, b*) has investigated cylindrical waves, finding exact solutions representing radiation from an infinite cylinder, and Trautman (1958), has used approximations to investigate the form of metric to be expected at great distances from an isolated radiating system. Dr Marder has kindly calculated for us the Riemann tensor for his radiation metric; he finds that for large r the dominant terms (in physical components with respect to a suitable chosen tetrad), are of just the same form as the plane wave canonical form (4·5).

Dr Trautman has given the Riemann tensor in his approximate theory of the radiation from an isolated system. Again, it is of the form (4·5). As one would expect, the dominant terms go like r^{-1} for large r. (The behaviour of the dominant terms in the cylindrical case is of doubtful physical significance, because the metric is not asymptotically Minkowskian.)

5. Conclusion

We have seen how plane gravitational waves can be defined by analogy with plane electromagnetic waves; the analogy depends on the symmetry properties of such waves. As remarked earlier, we would not expect to find plane gravitational waves in nature, except as the limiting forms at great distances of waves from a finite source. The plane-wave solutions, nevertheless, provide useful and interesting models for studying the properties of gravitational waves; it is to be hoped that their planeness is now seen to be beyond question—the existence of sandwich waves lying entirely between two surfaces which are planes in Minkowski space-time gives the strongest possible support to this contention. The existence of sandwich waves suggests also that gravitational waves may have no gravitational mass. A gravitational wave with gravitational mass would necessarily possess a 'tail'—a region behind it in which the effects of the wave-region made themselves felt as an ordinary (non-radiative) gravitational field. However, cylindrical waves do have tails, and so this may be a point at which the plane-wave model is not reliable, and the argument cannot be regarded as conclusive. This is yet one more problem whose solution will be within reach only when exact solutions representing waves from a finite source become available. In the absence of such solutions we have refrained from attempting to discuss energy transport by gravitational waves. It is clear from the relative acceleration acquired by test particles, as described in §§ 3 and 4, that energy is transferred to test particles by a plane wave, but this does not enable us to make quantitative assertions about energy transport in general. The present fluid state of the theory of the energy pseudo-tensor would not appear to justify a discussion of energy transport in terms of this concept.

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