Physics 530
Schwartzschild Geometry

In 1916, shortly after Einstein published his theory of gravity, Karl Schwartzschild gave the first solution to the equations. This solution was for both the empty space Einstein equations, and for the equations with constant coordinate density of energy. The key feature which made the equations solvable was his assumptions of spherical symmetry and time independence. In these notes I will solve the equations assuming only spherical independence– the time independence will come for free (Birkoff's theorem) in 4 dimensional empty spacetimes.

When one assumes spherical symmetry, one is assuming that the spacetime metric has three linearly independent Killing vectors ξ_a^A, $a = 1, 3$ which satisfy the "commutation relations"

$$\pounds_{\xi_a} \xi_b^A = \xi_c^A \tag{1}$$

where a, b, c are taken to be an even permutation of the numbers $1, 2, 3$ (an even permulation is one that can be obtained from the list $1, 2, 3$ by making an even number of swaps of adjacent pairs of numbers.). The tensor

$$\gamma^{AB} = \sum_a \xi_a^A \xi_a^B \tag{2}$$

has a zero Lie derivative.

$$\pounds_{\xi_b} \gamma^{AB} = \sum_a ((\pounds_{\xi_b}) \xi_a^A \xi_a^B + \xi_a^A \pounds_{\xi_b} \xi_a^B) \tag{3}$$

let us take $b = 1$ as an example, in which case we have

$$\pounds_{\xi_1} \gamma^{AB} = \xi_3^A \xi_2^B + \xi_2^A \xi_3^B - \xi_2^A \xi_3^B - \xi_3^A \xi_2^B = 0 \tag{4}$$

since $\pounds_{\xi_1} \xi_3^A = -\pounds_{\xi_3} \xi_1^A = \xi_2^A$.

Thus, if we take part of the metric to be $F(p)\gamma^{AB}$ where the function F obeys $\pounds_{\xi_a} F = 0$ this part of the metric will have zero Lie derivative.

We can define the θ, ϕ coordinates by choosing ξ_3^A to define the ϕ coordinate. First, since there are three Killing vectors, and they lie in a two dimensional space, at each point some linear combination of the three must equal zero. Choose those set points where $\xi_3^A = 0$ and are continuously related to each other as the coodinate $\theta = 0$. Define the ϕ surface as those points which that $\theta = 0$ point is dragged to by the Killing dragging along ξ_2^A. Label each point by the parameter along this Killing curve and call it θ. Then ϕ is defined by the parameter along the ξ_2^A Killing curves. θ labels each of these surfaces created by dragging the $\phi = 0$ point.

This results in the tensor components for γ^{AB}

$$\gamma^{\theta\theta} = 1 \tag{5}$$

$$\gamma^{\phi\phi} = \frac{1}{\sin^2(\theta)} \tag{6}$$

$$\gamma^{\theta\phi} = 0 \tag{7}$$

Define the third coordinate, r, to be such that the area of these two surfaces of symmetry are $4\pi r^2$. Choose the fourth coordinate u so that the "r" coordinate axis is a null vector. Since we have defined θ, ϕ, r purely using the symmetry, the u axis (θ, ϕ, r all constants) must be orthogonal to the surfaces of symmetry, since it it were not, the projection of that coordinate axis onto the surfaces of symmetry would pick out a unique direction in that two dimensional space, which would not be consistant with spherical symmetry.

Finally, we choose u so that the r coordinate axis (θ, ϕ, u all constant) is a null vector. There will in general be two ways to do this, and we choose u so that this is continuous throught the space. Again, having done this at one point on the two sphere of symmetry, one can define it everywhere on that sphere by rotation.

Again, since this choice is unique, the r coordinate axis must also be orthogonal to the surfaces of symmetry. We finally obtain the metric

$$ds^2 = \mathcal{V}(u,r)du^2 + 2\mathcal{U}(u,r)dudr - r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{8}$$

in these coordinates.

The inverse metric will be

$$g^{rr} = -\frac{\mathcal{V}}{\mathcal{U}^2} \tag{9}$$

$$g^{ur} = \frac{1}{\mathcal{U}} \tag{10}$$

$$g^{\theta\theta} = sin^2(\theta)g^{\phi\phi} = \frac{1}{r^2} \tag{11}$$

with all other components either given by these by symmetry or are zero.

Evaluation of the Christofel symbols

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \tag{12}$$

$$\Gamma^r{}_{\theta\theta} = \frac{1}{sin^2(\theta)}\Gamma^r{}_{\phi\phi} = -r \tag{13}$$

$$\Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{\phi r} = \frac{1}{r} \tag{14}$$

$$\Gamma^\theta{}_{\phi\phi} = -sin(\theta)cos(\theta) \tag{15}$$

$$\Gamma^\phi{}_{\phi\theta} = \frac{cos(\theta)}{sin(\theta)} \tag{16}$$

$$\Gamma^u{}_{uu} = -\frac{\partial_r \mathcal{V}}{\mathcal{U}} \tag{17}$$

$$\Gamma^r{}_{uu} = (-\frac{\mathcal{V}}{\mathcal{U}^2}(2\partial_u \mathcal{U} - \partial_r \mathcal{V}) + \frac{1}{\mathcal{U}}\partial_u \mathcal{V} \tag{18}$$

$$\Gamma^r{}_{ur} = \frac{\partial_u \mathcal{U}}{\mathcal{U}} \tag{19}$$

All the rest are 0.

We can now evaluate

$$R_{ij} = \partial_k \Gamma^k{}_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma_{kkl}\Gamma^l{}_{ij} - \Gamma^k{}_{il}\Gamma^l{}_{kj} \tag{20}$$

$$G_{ij} = R_{ij} - \frac{1}{2}g^{kl}R_{kl}g_{ij} \tag{21}$$

The key components are

$$G^u{}_r = 2\frac{\partial_r \mathcal{U}}{\mathcal{U}^2 r} \tag{22}$$

$$G^r{}_u = \frac{-2\mathcal{V}\partial_u \mathcal{U} + \mathcal{U}\partial_u \mathcal{V}}{\mathcal{U}^3 r} \tag{23}$$

$$G^r{}_r = -\frac{r\partial_r \mathcal{V} - \mathcal{U}^2 + \mathcal{V}}{\mathcal{U}^2 r^2} \tag{24}$$

From the first we have that U is independent of r and is thus a function of u only. Defining a new u coordinate by $\int |\mathcal{U}| du$ (and assuming that $\mathcal{U}$ is of one sign only) we set $\mathcal{U} = \pm 1$. From the second equation, we then get that $\mathcal{V}$ is also independent of u. From the third equation, we have

$$\mathcal{V} = 1 - \frac{\alpha}{r} \tag{25}$$

where α is an integration constant. If we take $t = u \pm \int \frac{dr}{1-\frac{\alpha}{r}} dr$ we get the metric

$$ds^2 = (1 - \frac{\alpha}{r})dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{26}$$

which for large r the correction of g_{tt} should be $1 + 2\Phi$ where Φ is the Newtonian potential $-G_N M/r$ (where . But this says that $\alpha = -2M(= \frac{2G_N M}{c^2})$ in units where G_N is also 1. Thus the metric is

$$ds^2 = (1 - \frac{2M}{r})du^2 \pm 2dudr - r^2 d\Omega^2 \tag{27}$$

where $d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2$.

The two signs are not two separate solutions, but the same solution in two different coordinates. If we start with

$$ds^2 = (1 - \frac{2M}{r})du^2 + 2dudr - r^2 d\Omega^2 \tag{28}$$

and we define $v = u + 2\int \frac{dr}{1-\frac{2M}{r}}$, the metric becomes

$$ds^2 = (1 - \frac{2M}{r})dv^2 - 2dvdr - r^2 d\Omega^2 \tag{29}$$

Ie, we get the above solution with the negative sign. We note that the coordinate transformation only works for $r > 2M$. As $r \to 2M$ from above, $r^* = \int \frac{dr}{1-\frac{2M}{r}}$ goes to infinity. Furthermore, if we look at a line $r = au$ say, this becomes the line $r = av - 2a(r - 2M\ln(\frac{r}{2M} - 1)$ in r, v coordinates, which says that v goes to infinity as r approaches $2M$, Ie, a perfectly nice line in u, r space becomes a line which approaches infinity assymptotically in v space. Similarly a line which goes through $r = 2M$ for finite v in v, r space, goes to $-\infty$ in u, r coordinates. Ie, the region $r < 2M, u$ is off at infinity in r, v space, and $r < 2M, v$ is at $-\infty$ in r, u space.

0.1 Schwartzschild

The above metrics are called metric in Eddington Finkelstein coordinates, because neither Edington, nor Finkelstein ever wrote down this solution. The found the coordinates that you would get if you defined $\tau = u + r$ or $\tau = v - r$ respectively. The above metrics in the last section (with constant M) were first written by Penrose in the 1960's. It is by far the simplest coordinates in which to solve for the spherical symmetric metric that I have found. Until Penrose, people tended to avoid null coordinates– ie coordinates for which one or more of the coordinate axes had null tangent vectors.

The first solution was by Schwartzscild in about about 5 months after Eistein published his field equations. Schwartzscild at that time was fighting on the eastern front against Russia, or rather was in hospital with the disease that finally killed him. Instead of using the θ coordinate, he used $\xi = cos(\theta)$ so that the angular part of the metric was $r^2 \frac{d\xi^2}{1-\xi^2} + (1-\xi^2)d\phi^2)$ so as to make the determinant of the angular part r^4. It was Droeste, a student, who wrote it in the θ, ϕ coordinates we now use.

Take the above metric and define $\tau = u + \int \frac{dr}{1-\frac{2M}{r}}$. The metric then becomes

$$ds^2 = (1 - \frac{2M}{r})d\tau^2 - \frac{dr^2}{(1-\frac{2M}{r})} - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2) \quad (30)$$

$$= dt^2 - dr^2 - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)) - \frac{2M}{r}dt^2 - \frac{2M}{r-2M}dr^2) \quad (31)$$

which is what is usually called the Schwartzscild metric. Einstein was astonished that Schwartzscild have found an exact solution so quickly.

0.2 Kruscal

One question is whether or not we can arrange coordinates such that $r = 2M$ is regular across both "horizons" (the "singularities" at $r = 2M$. Until the 1950s, there had been intense arguments about what the apparent singularities at $r = 2M$ meant. While a student C Lanzos had already in the early 20's pointed out that coordinate singularities could make the metric look pathalogical (thing on r=0 in polar coordinates, where the metric components go to 0), and desipte expolicit solutions (see Eddington and LeMaitre in the 20's) confusion reigned supreme about what those that "Schwartzschild singularity" meant.

Note that

$$1-\frac{2M}{r}=\frac{2M}{r}e^{(r-r^*)/rM}=\frac{2Me^{\frac{r}{2M}}}{r}e^{\frac{v-u}{4M}} \tag{32}$$

Thus, if we define $U=\int e^{-\frac{u}{4M}}$ and $V=\int e^{\frac{v}{4M}}dv$, Then the line $r=au$ becomes

$$r(1-2a)=v+4M\ln(r/2M-1)=4M(\ln(\frac{V}{4M})-\ln(r/2M-1) \tag{33}$$

$$=4M\ln(\frac{V}{2(\frac{r}{2M}-1)}) \tag{34}$$

or

$$V=2(\frac{r}{2M}-1)\,[\ln(r(1-2a))] \tag{35}$$

which is a regular curve through $r=2M$. similarly a curve through $r=2m$ in r,v space is a regular curve through $r=2M$ and $U=0$ in r,U coordinates.

Ie, the U,V coordinates seem to be mapping from the u,v coordinates which make the both horizons regular. This suggests that we choose our coordinates to be U,V rather than u,r or v,r. This leads to the metric

$$ds^2=\frac{4Me^{\frac{R}{4M}}}{R}dUdV-R(U,V)(d\theta^2+\sin^2(\theta)d\phi^2) \tag{36}$$

where $R(U,V)$ obeys

$$(\frac{R}{2M}-1)e^{\frac{R}{2M}}=-\frac{VU}{(4M)^2} \tag{37}$$

(which is a regular function at $V=0$ or $V=0$)

These coordinates are called the Kruskal coordinates. They were discovered by Martin Kruskal, who showed them to John Wheeler, asking if they might be of any interest. Wheeler then wrote up a paper, with Kruskal's name as the sole author, and sent it in to Physical Review. Kruscal was on sabatical in France that year, and the first he knew of the paper was when he received the galley proofs. He suspected that Wheeler had done this, and tried to get Wheeler to also put his name of the paper, and Wheeler refused, saying that Kruscal had done all the work.

(The same transformation was also found by Peter Szekeres and published in 1960 in a Hungarian Math journal.)

M.D. Kruskal, Phys. Rev. 119, 1743 (1960) and P. Szekeres, Publ. Math. Debrecen, 7 285 (1960)

0.3 Vaiyda metric

Looking back at the Einstein equations we note that the only place that the time derivative of V occurs is in the $G^u{}_r$ term. If we allow M to be a function of u, the only term in the stress energy tensor we need is $T^r{}_u$. All other components will be zero. If we raise the second index, since g^{uu} is zero, the only term will be $T^{rr} = -\frac{\partial_u M(u)}{\mathcal{U} r^2}$. But the r axis is a null vector. Thus the energy momentum tensor is that of a perfect fluid with no pressure, but with a flow of fluid along the null vector. Ie, this is a solution for either ingoing or outgoing null fluid. In order that the null fluid density be positive, we need dM/du to have the same sign as $\mathcal{U}$. If we take $M = M_0\Theta(u - u_0)$ where $\Theta(u)$ is the Heavyside step function, then the fluid source is a delta function shell. This would produce a metric which is just flat spacetime in the interior, and is the Schwartzschild solution outside.