Physics 530
Tensors

Tensors

A tensor is a linear function of vectors, either tangent vectors or cotangent vectors. The main problem with learning tensors is the notation and how to combine tensors into other tensors.

Thus, if we have a tensor T it means that it is a function of vectors– either tangent, cotangent (gradiant) or both. Thus in the usual way we would write a tensor as ,for example, a function of a tangent vector and a cotangent vector $T(V^A, W_B)$ To be a tensor, it would have to be a linear function (with real, or in some cases complex, result) of both arguments

$$T(\alpha V^A + \beta\tilde{V}^A, W_B) = \alpha T(V^A, W_B) + \beta T(\tilde{V}^A, W_B) \quad (1)$$
$$T(V^A, \alpha W_B + \beta W_B) = \alpha T(V^A, W_B) + \beta T(V^A, \tilde{W}_B) \quad (2)$$

Notationally, instead of writing out the tensor in terms of dummy arguments, one instead writes it out more compactly as ${T_A}^B$ so that in terms of the dummy arguments

$$T(V^A, W_B) \equiv {T_A}^B V^A W_B \quad (3)$$

Ie, the ${T_A}^B$ means that T is alinear function of tangent vector, and a cotangent vector. (Ie, the functional indices are the opposite of the argument). This is in sympathy with the inner product $U^A X_A$ between a tangent vector and a cotangent vector.

$U^A X_A$ is linear in X^A. and thus this inner product can be regarded as a linear function of the cotangent vector X_A and thus, the tangent vector U^A can be regarded as tensor with one argument, a cotangent vector. The inner product allows us to regard U^A as a tengent vector to some curve, or as a linear function of a cotangent vector. Which it is is a matter of context.

Similarly $U^A X_B$ is also a linear function of the tangent vector U^A. Thus X_B can be regarded as the gradiant of some function, or as a tensor function of a tangent vector.

Given a tensor, and a coordinate system (a set of functions x^i, $i = 0..D-1$ or $i = 1, D$ such that the equation $x^i(p) = x^i(p_0)$ has the unique solution of $p = p_0$. We then have the D gradient vectors at any point p of dx^i_A and the

D tangent vectors ∂_i^A whis are tangent to the curves through p_0 such that $\gamma(\lambda)p(x^{j\neq i(p_0)}, x^i = x^i(p_0)+\lambda$. Ie, this curve is the ith coordinate axis.

Then $dx^j(\gamma_i(\lambda)) = dx^i_A\partial^A_{\gamma_i} = \delta^i_i$ and if

$$df^A\partial_i^A = \partial_{x^i}f \tag{4}$$

$$df_A = \sum_i \partial_{x^i}f(p(x))dx^i_A \tag{5}$$

We can do the same for tensors

$$T_A{}^B = \sum_i jT_i{}^j dx^i_A\partial^B_{\gamma^i} \tag{6}$$

Tensor combinations

There are a number of combinations of tensors to make other tensors. Any combination of tensors which are linear functions of the arguments is a Tensor

a) Multiplying a tensor by a constant or a real function is a tensor

$$S^A{}_B = f(p)T^A{}_B \tag{7}$$

b) Adding two tensors of the same arguments

$$S^A{}_B = T^A{}_B + U^A{}_B \tag{8}$$

c)Mutiplying tensors of different arguments

$$S^A{}_{BC}{}^D = T^A{}_B U^D{}_c \tag{9}$$

d)Tracing

$$S_C = Y^A{}_{AC} \tag{10}$$

This is the trickiest of the lot, as the result has fewer arguments than the tensor it is made from. The A in the right hand side is no longer the placeholder for an argument. This operatiohn stems from the inner product of a tangent and cotangent vector. T^AU_B is a two argument tensor (rule c)But T^AU_A is a zero argument function, not a tensor. It is the inner product between a tangent and cotangent vector.

Thus if

$$Y^A{}_{BC} = sum_\alpha V^A_{(\alpha)} W^{(\alpha)}_B Q^{(\alpha)}_C \tag{11}$$

(α is just a label) then

$$Y^A{}_{AC} \equiv sum_\alpha V^A_{(\alpha)} W^{(\alpha)}_A Q^{(\alpha)}_C \tag{12}$$

Ie, the trace is simply a generalisation of the inner product of a tangent and cotangent vector.

Note that T^{AA} means nothing. It is not a tensor and is undefined. The components of a tensor are the tensorevaluated on the gradient of the coordinates and the tangent vectors to the coordinate axis curves.

$$T^i{}_{jk} = T^A{}_{BC} dx^i_A \partial^B_{\gamma^i} \partial^C_{\gamma^k} \tag{13}$$

The equaitons for the various tensor equations are identical, if you replace indices like A,B,C with indicees like i,j,k and where repeated indices in upper and lower postion are summed over.

Note that after Einstein, the summation over the indicees indicated a trace is not explicitly stated. It is just notationally understood that if, for componens an upper and lower index are the same, they are summed over. Thus

$$\sum_i Y^i{}_{ij} \equiv Y^i{}_{ij} \tag{14}$$

If one for some reason wants to sum over a repeated set of upper or lower indicees, then the summation symbol is mandatory. This is not a tensor operation $\sum_i Y^{ii}$ is not the component of a tensor and is "never" written as Y^{ii}. (Of course some people are weird and do it anyway. That is a way to making mistakes and creating confusion).

Metric (cont)

$g(p)_{AB}$ is the metric tensor at point p. It gives the length of tangent vector V^A by the evaluation of the tensor on the tangent vector V into both arguments of g

$$Length^2(V) = g_{AB} V^A V^B \tag{15}$$

Note that this is not a tensor. However, $g_{AB}V^B$ is a tensor with a single tangent vector argument. Thus it is equivalent to some cotangent vector $g_ABV^A \equiv U_B$ If g_{AB} is non-singular, (ie never maps a non-zero tangent vector onto the zero cotangent vector) then this also maps cotangent vectors onto unique tangent vectors (if $g_{AB}V^A = g_{AB}W^A$ then by linearity $g_{AB}(V^A - W^A) = 0$ when $V^A - W^A \neq 0$)

Thus there exists another tensor g^{AB} such that $V^C = g^{BC}U_B$ or $V_C = g_{AB}V^Bg^{BC} = (g_{AB}g^{BC})V^A$ for all vectors V^A Thus the tensor

$$\delta_A^C = g_{AB}g^{CB} \tag{16}$$

is the identity tensor. It converts a tangent vector into itself. Similarly it converts a cotangent vector into itself.Thus tensor g^{AB} is the inverse of the metric. It also acts as a metric for cotangent vectors, allowing us to define a length on cotangent vectors. Such a concept is a bit counterintuitive, since it is not clear what the length of a gradient means.

In component form, this is

$$\sum_k g_{ik}g^{jk} = \delta_i^j \tag{17}$$

Because of the importance of the metric to physics, it is a very special tensor. The equation $g_{AB}V^B = U_A$ is usually written as $g_{AB}V^B = V_A$ Ie one gives the same name to the cotangent vector as sociated via the metric with a tangent vector. They are clearly very diffe veryrent things– tangent vectors and cotangent vectors or gradients are physically very differnt objects. But the metric allows us to associate one with the other. Note that you have often heard that the gradient is a vector (arrow) which points in a direction perpendicular to the level surfaces of a function. This statement uses the above relation between tangent vectors and gradients.

Note that for any tangent vector lying in the level surface of a function, the inner product is

$$V^A df_A = \frac{d}{d\lambda} f(\gamma(\lambda)) = 0 \tag{18}$$

because by assumption the curve to which V is the tangent vector lies in a level surface, ie a surface where f(p) all have the same value. Ie $f(\gamma(\lambda)) = const$ and the derivative is 0. This means that if $U_A = (df)_A$ then

$$g_{AB}U^AV^B = U_BV^B = 0 \tag{19}$$

Ie, the tangent vector associated with the gradient is perpendicular (zero metric product) to any vector lying in the level surface.

Note that in Relativity, there exists non-zero tangent vectors and cotangent vectors such that their length is zero. Such vectors are called null-vectors. They are the tangent vectors to curves which are the paths of light rays. If one has a function whose gredient is a null vector, then the perpendicular tangent vector to that level surface of the function lies in the surface itself.

Since the metric associates a cotangent vector with every tangent vector, and vice-versa for the metric inverse, if we have a tensor ${S^A}_B$ say then the function is ${S^A}_B W_A V^B$. Now we can define a new tensor $\tilde{S}_{CB} X_C W_B = {S^A}_B (g_{AC} U^C) V^B$ where $W_A = g_{CA} U^C$ This new tensor $\tilde{S}_{CB}$ is written as $S_{CB} = {S^A}_B g_{AC}$.